Math 18.06 - Linear algebra

All readings are from the textbook "Introduction to linear algebra", 5–th edition, by Gilbert Strang

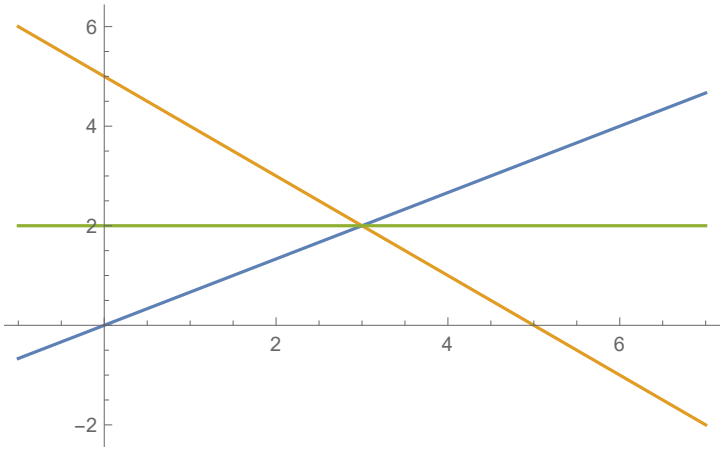
Lecture 1 (*September 4*) *Reading: sections 1.1–2.1*

In 18.02, you learned how to solve systems of linear equations:

$$\begin{cases} 2x - 3y = 0 \\ x + y = 5 \\ 2y = 4 \end{cases} \tag{1}$$

The high-school way to solve this system is to take y out from the last equation (since it is the more manageable of the three) and get $y = 2$. Then we plug this answer into either the first or the second equation, and we get $x = 3$. So the solution is $(x, y) = (3, 2)$.

There's also a geometric way to think about the system of equations (1). Linear equations, such as $2x - 3y = 0$, $x + y = 5$ or $2y = 4$, are equations of lines in the xy–plane. Specifically, they correspond to the blue, green and orange lines in the picture below:



Asking for a solution to the system of equations (1) is the same thing as asking for a point (x, y) which lies on all three of these lines, or in other words, the intersection of these lines. It's easy to see from the picture above that this point is $(x, y) = (3, 2)$.

Another point of view on the system of equations (1) is to rewrite it in matrix form:

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} \tag{2}$$

In this way, you solve for the quantity $\begin{bmatrix} x \\ y \end{bmatrix}$, which is thought of as the point (x, y) in the plane.
There are a number of ways to solve matrix equations like (2), the most general form of which is:

$$A\boldsymbol{v} = \boldsymbol{b} \tag{3}$$

where A is an $m \times n$ matrix, $\boldsymbol{v}$ is an $n \times 1$ vector, and $\boldsymbol{b}$ is a $m \times 1$ vector. In situations such as (3), usually A and $\boldsymbol{b}$ are given as part of the problem, while $\boldsymbol{v}$ is the unknown you have to solve for.

Before we delve into ways to solve (3), let us say a thing or two about vectors. We think of:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{4}$$

as a point in n–dimensional space. For example, 2–dimensional space is a plane (such as the surface of the paper, or a blackboard), while 3–dimensional space, is... well, ordinary space. Consider the following operations:

- given two vectors $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$, you may add them componentwise:

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

- given a vector $\boldsymbol{v}$ and a number α (called scalar), we can multiply them together:

$$\alpha\boldsymbol{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}$$

Combining the two operations above, we will say a **linear combination** of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is any expression of the form:

$$\alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} \alpha v_1 + \beta w_1 \\ \alpha v_2 + \beta w_2 \\ \vdots \\ \alpha v_n + \beta w_n \end{bmatrix}$$

The numbers α and β are called the **coefficients** of the linear combination. So we've seen the definition, but what does the concept of "linear combination" actually mean? Let's take an easy example:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad \text{linear combinations are } \alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} \alpha \\ 0 \\ \beta \end{bmatrix}$$

As α and β run over all possible numbers, the right-hand side runs over all vectors in the xz–plane (which in turn is a subset of xyz–space). Given that $\boldsymbol{v}$ is the unit vector on the x–axis and $\boldsymbol{w}$ is the unit vector on the z–axis, the principle here is the following.

Fact 1. *Almost* [1] *any two $n \times 1$ vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ trace out a 2–dimensional plane in n–dimensional space. Any vector in the aforementioned plane is a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$, and vice-versa.*

Example 1. *Let's draw the plane traced out by the vectors* $\boldsymbol{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ *and* $\boldsymbol{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

To do this, mark $\boldsymbol{v}$ and $\boldsymbol{w}$ as lattice points in space. Then trace out the lines passing through the origin and either the point $\boldsymbol{v}$ or $\boldsymbol{w}$. Finally, trace out the 2–plane spanned by these two lines.

We can use the language of linear combinations to make sense of systems of equations such as (3). Let's look at the specific example (2). Because:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then formula (2) reads:

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \cdot \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$

Therefore, solving for (x, y) is the same thing as finding a certain linear combination of the vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ (which are nothing but the columns of the 3×2 matrix in the left-hand side of (2)) that produces the vector $\begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$. Such a linear combination only exists if the latter vector lies in the plane spanned out by the former vectors, which is indeed the case in our example. In general:

Fact 2. *The equation* $A\boldsymbol{v} := \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} =: \boldsymbol{b}$ *is equivalent to:*

$$v_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots + v_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \tag{5}$$

This precisely asks which linear combinations of the columns of the matrix A are equal to the vector $\boldsymbol{b}$.

[1] The word "almost" requires us to exclude the case when $\boldsymbol{v}$ and $\boldsymbol{w}$ are collinear, i.e. on the same line passing through the origin; we will discuss this issue further when introducing the notion of linear independence

There is another operation that one can perform on two vectors, the **dot product**:

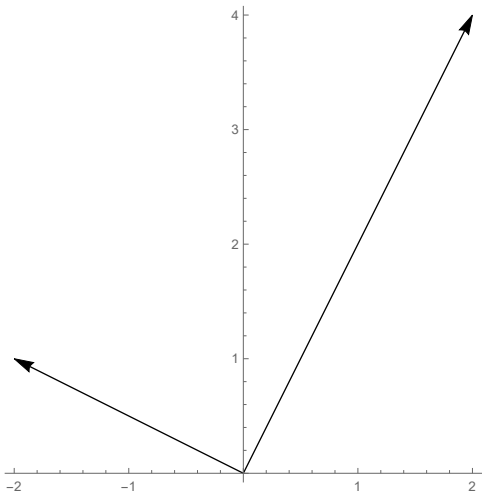
$$\text{if } \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \quad \text{then} \quad \boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + ... + v_n w_n$$

VERY IMPORTANT: the dot product of two vectors is a number (a.k.a. scalar). There is no way to multiply two vectors in such a way as to obtain a vector (with the notable exception of the cross-product for vectors with $n = 3$ components, but that's another story).

The dot product allows us to describe the angle θ between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. Recall that:

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0 \qquad \text{means that} \qquad \theta = 90^\circ$$

For example, take $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$:



The angle between these two vectors is 90°, and indeed $\boldsymbol{v} \cdot \boldsymbol{w} = (-2) \cdot 2 + 1 \cdot 4 = 0$. But the dot product describes arbitrary angles, not just right angles. The general formula is:

$$\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}||\ ||\boldsymbol{w}||} \tag{6}$$

where the **length** of a vector is given by:

$$||\boldsymbol{v}|| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + v_2^2 + ... + v_n^2} \tag{7}$$

So if you want to compute the angle between two lines, apply formula (6) with $\boldsymbol{v}$ and $\boldsymbol{w}$ being the unit (i.e. length 1) vectors corresponding to those lines. Then the formula in question precisely says that the cosine of the angle is equal to the dot product of unit vectors.

Lecture 2 (*September 6*) *Reading: sections 2.2–2.3*

The basic method for solving systems of equations is **Gaussian elimination**. For instance:

$$\begin{cases} x - y + 2z = 1 \\ -2x + 2y - 3z = -1 \\ -3x - y + 2z = -3 \end{cases} \qquad \text{or, equivalently,} \quad \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} \tag{8}$$

The 3×3 matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix}$ is called the **coefficient matrix**, while $\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the unknown vector. Geometrically, the system (8) consists of three linear equations in xyz–space. Each of these equations determines a 2–dimensional plane in 3–dimensional space, so the system of equations computes the coordinates (x, y, z) of the intersection point of three planes.

Gaussian elimination is the following algorithm. Consider the **augmented matrix**:

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ \boxed{-2} & 2 & -3 & -1 \\ \boxed{-3} & -1 & 2 & -3 \end{array}\right] \tag{9}$$

obtained by tacking the right-hand side of (8) onto the right of the matrix A. The **pivot** of each row (marked by a box) refers to the leftmost non-zero entry on that row. The game is the following: by performing **row eliminations** (namely adding any multiple of row j to row i, for various $i > j$), your goal is to ensure that the pivot of the first row is strictly to the left of the pivot on the second row, which in turn is strictly to the left of the pivot on the third row ... etc.

The particular row eliminations you have to do are imposed upon by the following algorithm. Start from the pivot on the first row. I claim that there's a single choice of row eliminations which will ensure that the entries below the pivot on the first row will be 0. This is because:

$$a \cdot 1 + (-2) = 0 \text{ implies } a = 2$$
$$b \cdot 1 + (-3) = 0 \text{ implies } b = 3$$

so we must add 2 times the first row to the second row and 3 times the first row to the third row, in order to achieve our stated goal of having the entries below the pivot be 0.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2+2\cdot 1 & 2+2\cdot(-1) & -3+2\cdot 2 & -1+2\cdot 1 \\ -3+3\cdot 1 & -1+3\cdot(-1) & 2+3\cdot 2 & -3+3\cdot 1 \end{array}\right] = \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & \boxed{-4} & 8 & 0 \end{array}\right]$$

The next step is to find the the pivot of the second row, and repeat the process. But wait! In the example above, the pivot on the second row is to the right of the pivot on the third row. Whenever this happens, there's no row elimination which will put the pivots in the right order (namely the pivot on the second row to the left of the pivot on the third row). So we allow ourselves to make an extra move: **row exchanges**. Specifically, we simply swap the second and third rows. We obtain:

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{-4} & 8 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \tag{10}$$

This augmented matrix is in **row echelon form**, which is the purpose of Gaussian elimination: all the pivots are non-zero, and each pivot is strictly to the right of the one on the row above.

Remark. *There's a variant of this algorithm, called Gauss-Jordan elimination, which entails doing two additional steps. Firstly, multiply all the rows of the augmented matrix by appropriate constants,*

so that all the pivots are equal to 1:

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

Then, we do some further row operations so that all the entries above the pivots are 0. In the case of the matrix at hand, we add 2 *times the third row to the second row:*

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

and then we add -2 *times the third row and* 1 *times the second row to the first row:*

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

The output of Gauss-Jordan elimination is said to be in ***reduced row echelon form****, i.e. a matrix which is both in row echelon form, and satisfies the two conditions underlined above.*

The augmented matrix (10) corresponds to the following system of linear equations:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{or, equivalently,} \quad \begin{cases} x - y + 2z = 1 \\ -4y + 8z = 0 \\ z = 1 \end{cases} \tag{11}$$

and the great thing about Gaussian elimination is that this system of equations is actually equivalent to (8): any solution (x, y, z) of (8) is also a solution of (11), and vice versa.

So why did we go through the trouble of Gaussian (or Gauss-Jordan) elimination? This is because systems where the coefficient matrix is in (reduced) row echelon form are really easy to solve by hand, by the following procedure (for which the technical term is **back substitution**): the third equation in (11) gives you $z = 1$, which you can plug into the second equation to get $y = 2$, which you can plug into the first equation to get $x = 1$. So the solution is $(x, y, z) = (1, 2, 1)$.

Remark. *If the row echelon form of a matrix A has a full row of zeroes (i.e. no pivot on the last row), then the matrix A is called* ***singular****. In this case, the system* (11) *may not have any solutions or may have more solutions than expected. Otherwise, the matrix is called* ***non-singular****.*

So what is going on behind this method? Back substitution provided such a quick answer because:

$$\boxed{U\boldsymbol{v} = \boldsymbol{c}} \tag{12}$$

is a very easy system to solve if U is in row echelon form. Indeed, you solve for the last component of the vector $\boldsymbol{v}$ first, then for the next-to-last one, then for the next-to-next-to-last one etc. If you need to solve a system of equations for a general matrix A, then Gaussian elimination precisely provides a recipe for putting A in row echelon form, at the cost of performing the following row operations (for any $i > j$):

- Adding the j–th row times λ to the i–th row of A is achieved by multiplying the latter on the left with an **elimination matrix**:

$$A \rightsquigarrow E_{ij}^{(\lambda)} A \qquad \text{where} \qquad E_{ij}^{(\lambda)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  The entry λ is on the i–th row and j–th column (in the formula above, $i = 4$ and $j = 2$). [2]

- Switching the i–th and j–th rows of A is achieved by multiplying the latter on the left with a **permutation matrix**:

$$A \rightsquigarrow P_{ij} A \qquad \text{where} \qquad P_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  Explicitly, P_{ij} has coefficients 1 on positions (i, j), (j, i) and (k, k) for all $k \neq i, j$, and all other coefficients of P_{ij} are 0 (in the formula above, $i = 3$ and $j = 4$).

- If you wish to put the matrix in reduced row echelon (as opposed to just row echelon) form, you will also need to multiply the i–th row of A by various constants λ, which is achieved by multiplying A by a **diagonal** matrix:

$$A \rightsquigarrow D_i^{(\lambda)} A \qquad \text{where} \qquad D_i^{(\lambda)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  The entry λ is on the i–th row (in the formula above, $i = 2$).

So let's see what these matrices amount to in the example of (8). We will forget about the 3×4 augmented matrix $[A|\boldsymbol{b}]$ and just discuss Gaussian elimination on the 3×3 matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix}$$

and the first thing we did was to add 2 times the first row to the second row:

$$E_{21}^{(2)} A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ -3 & -1 & 2 \end{bmatrix}$$

[2] If you're doing Gauss-Jordan elimination, you will also need the elimination matrices $E_{ij}^{(\lambda)}$ for $i < j$.

The next thing we did was to add 3 times the first row to the third row:

$$E_{31}^{(3)}E_{21}^{(2)}A = \begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1&-1&2\\0&0&1\\-3&-1&2\end{bmatrix} = \begin{bmatrix}1&-1&2\\0&0&1\\0&-4&8\end{bmatrix}$$

Then we exchanged the second and third rows:

$$P_{23}E_{31}^{(3)}E_{21}^{(2)}A = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&-1&2\\0&0&1\\0&-4&8\end{bmatrix} = \begin{bmatrix}1&-1&2\\0&-4&8\\0&0&1\end{bmatrix} =: U \tag{13}$$

and we have achieved precisely the row echelon form matrix of (11).

Remark. *Suppose we wish to go further, and do Gauss-Jordan elimination. Then we need to make all the pivots 1, and to this end we must multiply the second row by $-1/4$:*

$$D_2^{\left(-\frac{1}{4}\right)}U = \begin{bmatrix}1&0&0\\0&-\frac{1}{4}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&-1&2\\0&-4&8\\0&0&1\end{bmatrix} = \begin{bmatrix}1&-1&2\\0&1&-2\\0&0&1\end{bmatrix}$$

The pivots are the 1's on the diagonal. To put U in reduced row echelon form, we need to kill all the non-zero entries above the 1's. This is achieved by adding 2 times the third row to the second row, then adding -2 times the third row to the first row, then adding the second row to the first:

$$E_{12}^{(1)}E_{13}^{(-2)}E_{23}^{(2)}D_2^{\left(-\frac{1}{4}\right)}U = \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&-2\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} =: R$$

The matrix R is now in reduced row echelon form.

Very importantly, this procedure also allows us to convert the system of equations (8) into a system of the form (12). Specifically, take the equality (8) and multiply it on the left with the particular product of elimination/permutation/diagonal matrices which appears in (13). We obtain:

$$\underbrace{P_{23}E_{31}^{(3)}E_{21}^{(2)}A}_{U}\boldsymbol{v} = P_{23}E_{31}^{(3)}E_{21}^{(2)}\boldsymbol{b} \tag{14}$$

If we let:

$$\boldsymbol{c} := P_{23}E_{31}^{(3)}E_{21}^{(2)}\boldsymbol{b} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\-1\\-3\end{bmatrix} =$$

$$= \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1\\1\\-3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

then formula (14) reads precisely (12), which establishes the following important principle:

Fact 3. *If:*

$$\boxed{A = LU \text{ and } \boldsymbol{b} = L\boldsymbol{c}}$$

for some matrix L (which in practice will be a product of elimination/permutation/diagonal matrices), then the system $A\boldsymbol{v} = \boldsymbol{b}$ is equivalent (i.e. has the same solution $\boldsymbol{v}$) as the system $U\boldsymbol{v} = \boldsymbol{c}$.

Since we have already seen that it is easy to solve a system of equations where the matrix U is in row echelon form, this gives a pretty general algorithm for solving general systems of equations.

Remark. *Gaussian (or Gauss-Jordan) elimination can be applied to any $m \times n$ matrix, producing a matrix in row (or reduced row) echelon form. The fact that in the first part of this lecture, we applied it to the augmented matrix $[A|\boldsymbol{b}]$ is simply because this is what you're supposed to do if you want Gaussian elimination to help you solve systems of equations. But as we have seen in the second part of this lecture, the method of Gaussian elimination can be applied to the matrix A itself.*

Lecture 3 (*September 9*) *Reading: sections 2.4–2.5*

Let's delve a little deeper into matrix multiplication. The key thing is the following:

$$\Big(m \times n \textbf{ matrix}\Big)\Big(n \times p \textbf{ matrix}\Big) = \Big(m \times p \textbf{ matrix}\Big) \tag{15}$$

AB only makes sense if the number of columns of A equals the number of rows of B.

Assume A is an $m \times n$ matrix with entries a_{ij} and that B is an $n \times p$ matrix with entries b_{ij}. Then the product $C = AB$ is a $m \times p$ matrix, whose entry on the i–th row and j–th column is equal to:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \tag{16}$$

Here's how to see the same computation by looking at the matrices themselves:

$$\underbrace{\begin{bmatrix} * & * & * & * & * \\ * & * & c_{ij} & * & * \\ * & * & * & * & * \end{bmatrix}}_{AB} = \underbrace{\begin{bmatrix} * & * & * & * \\ a_{i1} & a_{i2} & \dots & a_{in} \\ * & * & * & * \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} * & * & b_{1j} & * & * \\ * & * & b_{2j} & * & * \\ * & * & \dots & * & * \\ * & * & b_{nj} & * & * \end{bmatrix}}_{B}$$

(in the example above, $m = 3$, $n = 4$, $p = 5$, $i = 2$, $j = 3$). The i–th row of the matrix A and the j–th column of the matrix B are vectors of the same size, namely n. Formula (16) precisely says that the (i, j)–entry in the matrix AB is the **dot product** of these two vectors.

What if we wanted to look at the entire i–th row of the matrix AB?

$$\underbrace{\begin{bmatrix} * & * & * & * & * \\ c_{i1} & c_{i2} & \dots & \dots & c_{ip} \\ * & * & * & * & * \end{bmatrix}}_{C=AB} = \underbrace{\begin{bmatrix} * & * & * & * \\ a_{i1} & a_{i2} & \dots & a_{in} \\ * & * & * & * \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \dots & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & \dots & b_{np} \end{bmatrix}}_{B}$$

The entire i–th row of AB is a linear combination of the rows of B, according to the rule:

$$\begin{bmatrix} c_{i1} & c_{i2} & \dots & \dots & c_{ip} \end{bmatrix} = a_{i1} \begin{bmatrix} b_{11} & b_{12} & \dots & \dots & b_{1p} \end{bmatrix} + ... + a_{in} \begin{bmatrix} b_{n1} & b_{n2} & \dots & \dots & b_{np} \end{bmatrix}$$

Similarly, we may ask about the entire j–th column of the matrix AB:

$$\underbrace{\begin{bmatrix} * & * & c_{1j} & * & * \\ & & \vdots & & \\ * & * & c_{mj} & * & * \end{bmatrix}}_{C=AB} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} * & * & b_{1j} & * & * \\ * & * & b_{2j} & * & * \\ & & \vdots & & \\ * & * & b_{nj} & * & * \end{bmatrix}}_{B}$$

The entire j–th column of AB is a linear combination of the columns of A, according to the rule:

$$\begin{bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} b_{1j} + \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} b_{2j} + ... + \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} b_{nj}$$

The two computations we have just seen are particular instances of **block multiplication** of matrices. Explicitly, we say that matrices A and B have block decompositions:

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

if A_{ij} is an $m_i \times n_j$ matrix, and B_{jk} is a $n_j \times p_k$ matrix for all $i, j, k \in \{1, 2\}$ and some integers $m_1, m_2, n_1, n_2, p_1, p_2$. **Very important:** the sizes of the blocks in a block decompositions always match at the boundaries (for example, the number of rows of A_{11} is the same as the number of rows of A_{12}, and the number of columns of A_{11} is the same as the number of columns of A_{21}). Then the product matrix AB also has a block decomposition:

$$AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right] \tag{17}$$

where each of the four blocks in (17) is matrix multiplication in its own right (the sizes of these blocks are $m_i \times p_k$). Thus, you may reduce the multiplication of the bigger matrices A and B to the multiplication of the smaller matrices A_{ij} and B_{jk}. This allows you to recursively reduce matrix multiplication to the smallest blocks, namely 1×1, which are simply individual matrix entries.

Matrix multiplication satisfies many important rules, such as:

- associativity: $A(BC) = (AB)C$
- distributivity: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
- unit element: $AI = IA = A$, where $I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ is called the unit (or identity) matrix

Moreover, given a square matrix A, we can form its powers:

$$A^p = \underbrace{AA...A}_{p \text{ times}}$$

Just like with numbers, we set:

$$A^0 = I = \text{the identity matrix}$$

but can we talk about negative powers? This is the point of the following definition.

Definition 1. *Given a square matrix A, its **inverse** is a matrix A^{-1} with the property that:*

$$AA^{-1} = A^{-1}A = I \tag{18}$$

Not all square matrices have inverses (those which do are called **non-singular** and those which do not are called **singular**) but if an inverse exists, it is unique. For example, consider the matrix:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$$

If it had an inverse $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we would have:

$$\begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{19}$$

As we have seen, this implies that the columns of the identity matrix, namely:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{20}$$

are both linear combinations of the columns of A, namely:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 \\ 10 \end{bmatrix} \tag{21}$$

But this is clearly impossible, as the vectors (21) are both on the line of slope 2 in the plane, and therefore so is any linear combination of them. Meanwhile, the vectors (20) are not confined to this line, so we conclude that formula (19) cannot hold, and therefore A is singular.

The argument we have just given can be combined with Gauss-Jordan elimination to provide an algorithm for computing the inverse of a matrix A. Start by constructing the augmented matrix:

$$\left[\begin{array}{c|c} A & I \end{array}\right] \tag{22}$$

where A and I are both $n \times n$. If A is non-singular, then Gauss-Jordan elimination will output:

$$\left[\begin{array}{c|c} I & B \end{array}\right] \tag{23}$$

for some matrix B. Now how does Gauss-Jordan elimination get you from (22) to (23)? We have seen last time that it does this by multiplying (22) on the left with another matrix C:

$$C \left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{c|c} I & B \end{array}\right]$$

But from the rules of multiplying block matrices, this is equivalent to the two matrix equalities:

$$CA = I \quad \text{and} \quad CI = B$$

Therefore, $C = CI = B$, hence $BA = I$, hence B is the sought-for inverse of A. So the right-most block of (23), which is produced by Gauss-Jordan elimination, is precisely the inverse of A.

Being able to calculate inverses of matrices is a very helpful skill, since they allow you to solve square systems of equations (i.e. those systems where the number of equations is equal to the number of unknowns). Specifically, if you have a system $A\boldsymbol{v} = \boldsymbol{b}$ and you multiply it on the left by A^{-1}, then you obtain:

$$\boldsymbol{v} = A^{-1}\boldsymbol{b} \tag{24}$$

which is precisely the same as solving for the unknown vector $\boldsymbol{v}$. A useful tool in computing matrices is the following formula:

$$(AB)^{-1} = B^{-1}A^{-1} \tag{25}$$

which says that the inverse of a product (of square matrices) is the product of the inverses, but taken in reverse order. This is very meaningful, since the order in which we multiply matrices is important (a fancy way of saying this is that matrix multiplication is not commutative). Finally, a nice consequence of being able to define inverse matrices is that the expression:

$$A^n \tag{26}$$

now makes sense for any integer n, if A is a non-singular square matrix. If n is negative, then you just define (26) as the $-n$ th power of the inverse matrix A^{-1}. This is just the basic requirement that the usual formulas for powers:

$$A^m A^n = A^{m+n} \qquad \text{and} \qquad (A^m)^n = A^{mn} \tag{27}$$

continue to hold for matrices.

Lecture 4 (*September 11*) *Reading: section 2.6*

Let us return to Gaussian elimination, in the particular case of a square $n \times n$ matrix. If the algorithm doesn't require any rows to be exchanged (which is the case for most, but not all matrices), then the algorithm goes as follows:

- first step: add a multiple λ_{21} of the first row to the second, then a multiple λ_{31} of the first row to the third, ... then a multiple λ_{n1} of the first row to the n–th row
- second step: add a multiple λ_{32} of the second row to the third, then a multiple λ_{42} of the second row to the fourth row, ... then a multiple λ_{n2} of the second row to the n–th row
- ...

- $(n-2)$–th step: add a multiple $\lambda_{n-1,n-2}$ of the $(n-2)$–th row to the $(n-1)$–th row, then add a multiple $\lambda_{n,n-2}$ of the $(n-2)$–th row to the n–th row

- $(n-1)$–th step: add a multiple $\lambda_{n,n-1}$ of the $(n-1)$–th row to the n–th row.

As we have seen in Lecture 2, each step in the algorithm above involves multiplying A on the left by an elimination matrix. As a formula, this states that:

$$\underbrace{E_{n,n-1}^{(\lambda_{n,n-1})}}_{(n-1)\text{–th step}} \underbrace{E_{n,n-2}^{(\lambda_{n,n-2})} E_{n-1,n-2}^{(\lambda_{n-1,n-2})}}_{(n-2)\text{–th step}} \underbrace{E_{n,n-3}^{(\lambda_{n,n-3})} E_{n-1,n-3}^{(\lambda_{n-1,n-3})} E_{n-2,n-3}^{(\lambda_{n-2,n-3})}}_{(n-3)\text{–th step}} \ldots$$
$$\ldots \underbrace{E_{n2}^{(\lambda_{n2})} \ldots E_{42}^{(\lambda_{42})} E_{32}^{(\lambda_{32})}}_{\text{second step}} \underbrace{E_{n1}^{(\lambda_{n1})} \ldots E_{31}^{(\lambda_{31})} E_{21}^{(\lambda_{21})}}_{\text{first step}} A = U \quad (28)$$

where U is a matrix in row echelon form. Because we are in the square matrix case, U is also a square matrix, so row echelon form is the same thing as upper triangular.

Fact 4. *Each elimination matrix $E_{ij}^{(\lambda)}$ is a lower triangular matrix, and its inverse is given by:*

$$\left(E_{ij}^{(\lambda)}\right)^{-1} = E_{ij}^{(-\lambda)} \quad (29)$$

Indeed, let's establish formula (29) in the case of 2×2 matrices (so $i = 2$, $j = 1$):

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}$$

which is an easy computation to prove by hand. The proof of (29) for general $n \times n$ matrices is similar, and you can think about it, if you'd like.

Using formula (29), let's convert (28) into a formula for A. The first step to doing so is to multiply equation (28) on the left by the inverse of $E_{n,n-1}^{(\lambda_{n,n-1})}$, which according to (29) is precisely $E_{n,n-1}^{(-\lambda_{n,n-1})}$:

$$\left(E_{n,n-2}^{(\lambda_{n,n-2})} E_{n-1,n-2}^{(\lambda_{n-1,n-2})}\right)\left(E_{n,n-3}^{(\lambda_{n,n-3})} E_{n-1,n-3}^{(\lambda_{n-1,n-3})} E_{n-2,n-3}^{(\lambda_{n-2,n-3})}\right)\ldots\left(E_{n1}^{(\lambda_{n1})} \ldots E_{31}^{(\lambda_{31})} E_{21}^{(\lambda_{21})}\right) A =$$
$$= \left(E_{n,n-1}^{(\lambda_{n,n-1})}\right)^{-1} U = E_{n,n-1}^{(-\lambda_{n,n-1})} U$$

Then multiply the equation above on the left by the inverse of $E_{n,n-2}^{(\lambda_{n,n-2})} E_{n-1,n-2}^{(\lambda_{n-1,n-2})}$, which according to (25) and (29) is precisely $E_{n-1,n-2}^{(-\lambda_{n-1,n-2})} E_{n,n-2}^{(-\lambda_{n,n-2})}$. We obtain:

$$\left(E_{n,n-3}^{(\lambda_{n,n-3})} E_{n-1,n-3}^{(\lambda_{n-1,n-3})} E_{n-2,n-3}^{(\lambda_{n-2,n-3})}\right)\ldots\left(E_{n1}^{(\lambda_{n1})} \ldots E_{31}^{(\lambda_{31})} E_{21}^{(\lambda_{21})}\right) A =$$
$$= \left(E_{n,n-2}^{(\lambda_{n,n-2})} E_{n-1,n-2}^{(\lambda_{n-1,n-2})}\right)^{-1} \left(E_{n,n-1}^{(-\lambda_{n,n-1})}\right) U = \left(E_{n-1,n-2}^{(-\lambda_{n-1,n-2})} E_{n,n-2}^{(-\lambda_{n,n-2})}\right)\left(E_{n,n-1}^{(-\lambda_{n,n-1})}\right) U$$

We repeat this multiplication procedure $n-3$ times, and we arrive at the formula:

$$A = \left(E_{21}^{(-\lambda_{21})} \ldots E_{n1}^{(-\lambda_{n1})}\right)\ldots\left(E_{n-1,n-2}^{(-\lambda_{n-1,n-2})} E_{n,n-2}^{(-\lambda_{n,n-2})}\right)\left(E_{n,n-1}^{(-\lambda_{n,n-1})}\right) U \quad (30)$$

Now here's the kicker: each $E_{ij}^{(\lambda)}$ is a lower triangular matrix, namely one which has zeroes above the diagonal, and moreover the diagonal entries are all 1. It is easy to see that a product of lower triangular matrices is still lower triangular, for example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$$

Therefore, the product of elimination matrices in (30), namely:

$$L = \left(E_{21}^{(-\lambda_{21})} ... E_{n1}^{(-\lambda_{n1})}\right) ... \left(E_{n-1,n-2}^{(-\lambda_{n-1,n-2})} E_{n,n-2}^{(-\lambda_{n,n-2})}\right) \left(E_{n,n-1}^{(-\lambda_{n,n-1})}\right) \tag{31}$$

is still lower triangular, and moreover its diagonal entries are all 1. We conclude that:

$$\boxed{A = LU} \tag{32}$$

where L is a lower triangular matrix and U is an upper triangular matrix. Therefore, we summarize this whole discussion by saying that: **Gaussian elimination on a square matrix is equivalent to writing it as a product of a lower and an upper triangular matrix** (assuming there are no row exchanges). Moreover, if A is non-singular, then the matrices L and U in (32) are unique.

Let's do an example, namely Gaussian elimination of the matrix:

$$A = \begin{bmatrix} 2 & 4 & 1 \\ -4 & -5 & 0 \\ -2 & 5 & 6 \end{bmatrix}$$

First, let's add 2 times the first row to the second row, and we get:

$$E_{21}^{(2)} A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ -4 & -5 & 0 \\ -2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ -2 & 5 & 6 \end{bmatrix}$$

Then, let us add 1 times the first row to the third row, and we get:

$$E_{31}^{(1)} E_{21}^{(2)} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ -2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 9 & 7 \end{bmatrix}$$

Finally, we add -3 times the second row to the third row, and we get:

$$E_{32}^{(-3)} E_{31}^{(1)} E_{21}^{(2)} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 9 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} =: U \tag{33}$$

Since the matrix on the right is upper triangular, hence in row echelon form, we are done with Gaussian elimination! To write A in the form (32), we need to isolate A in the left-hand side. To do so, we must successively multiply the equation above with the inverses of the elimination matrices $E_{32}^{(-3)}$, $E_{31}^{(1)}$ and $E_{21}^{(2)}$. We have:

$$E_{31}^{(1)} E_{21}^{(2)} A = E_{32}^{(3)} U$$

$$E_{21}^{(2)} A = E_{31}^{(-1)} E_{32}^{(3)} U$$

$$A = E_{21}^{(-2)} E_{31}^{(-1)} E_{32}^{(3)} U$$

If we set:

$$L = E_{21}^{(-2)} E_{31}^{(-1)} E_{32}^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \tag{34}$$

then we conclude that $A = LU$ holds with U and L given by (33) and (34), respectively. Note that L has 1's on the diagonal, while U does not. If you'd like, you can write U as a diagonal matrix times an upper triangular matrix with 1's on the diagonal, for example:

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

Basically, this is done by writing $U = D\tilde{U}$, where D is the diagonal matrix with the same diagonal entries as the upper triangular matrix U, and $\tilde{U}$ is upper triangular with 1's on the diagonal.

Definition 2. *The LU factorization of a matrix A consists of the unique matrices U and L (the former upper and the latter lower triangular with 1's on the diagonal) such that $A = LU$ holds.*

Similarly, the LDU factorization of a matrix A consists of the unique matrices U, D and L (where D is diagonal, while U and L are upper/lower triangular with 1's on the diagonal) such that $A = LDU$.

Note that if you have the LU factorization, you can easily get the LDU factorization, essentially by procedure (35). Let's end with a remark: suppose you do Gaussian elimination on a square matrix A, and you bring it in the form (30). Now you may be taken aback by the task of actually having to perform the multiplication (31) in order to compute L. However, a great fact about lower triangular matrices is the following formula:

$$\left(E_{21}^{(-\lambda_{21})} \ldots E_{n1}^{(-\lambda_{n1})}\right) \ldots \left(E_{n-1,n-2}^{(-\lambda_{n-1,n-2})} E_{n,n-2}^{(-\lambda_{n,n-2})}\right) \left(E_{n,n-1}^{(-\lambda_{n,n-1})}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\lambda_{21} & 1 & 0 & 0 & 0 \\ \vdots & \vdots & 1 & 0 & 0 \\ -\lambda_{n-1,1} & -\lambda_{n-1,2} & \ldots & 1 & 0 \\ -\lambda_{n1} & -\lambda_{n2} & \ldots & -\lambda_{n,n-1} & 1 \end{pmatrix}$$

Note that this formula only holds because of the specific order in which the matrices $E_{ij}^{(-\lambda)}$ were multiplied in the left-hand side. If you had multiplied the matrices in (almost) any other order, you would not have obtained a result as nice as the right-hand side.

Lecture 5 (*September 13*) *Reading: section 2.7*

A **permutation matrix** is a square matrix which has a single 1 on each row and column, and 0 everywhere else. For example, the following is a 4×4 permutation matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{36}$$

For all $i \in \{1, ..., n\}$, look at the single 1 on the i–th row, and let $\sigma(i) \in \{1, ..., n\}$ be the column where that 1 lies. By the very definition of a permutation matrix, the numbers $\sigma(1), ..., \sigma(n)$ are all different, hence they determine a permutation of the numbers $1, ..., n$. For example, the permutation corresponding to the matrix (36) is $(2, 1, 4, 3)$. This is the reason why permutation matrices are called this way, because they are in one-to-one correspondence with usual permutations. Therefore:

$$\text{the number of } n \times n \text{ permutation matrices is } n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot (n-1) \cdot n \tag{37}$$

The particular matrix P_{ij} we encountered in Lecture 2 (when discussing the matrix meaning of the various steps in Gaussian elimination) corresponds to the permutation of the numbers $1, ..., n$ that switches i and j, but keeps all other numbers unchanged.

Suppose you do Gaussian elimination on a square matrix A, always taking care that the pivots are in the right place before doing elimination. For example, we may have:

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

Since the pivot on the first row is to the right of the pivots on the other two rows, let us remedy the situation by exchanging the roles of the first and third rows. As we have seen in Lecture 2, this is achieved by multiplying A on the left with the permutation matrix P_{13}:

$$P_{13}A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Then we do elimination, by subtracting the first row from the second one:

$$E_{21}^{(-1)} P_{13} A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{bmatrix} =: U$$

and the right-hand side is not in row echelon form, i.e. upper triangular. We may move any product of E's to the right-hand side by multiplying the equation above with their inverses, thus obtaining:

$$P_{13}A = E_{21}^{(1)} U$$

Since $E_{21}^{(1)}$ is a lower triangular matrix and P_{13} is a permutation matrix, this is a particular example of the following general formula, which holds for any non-singular matrix A:

$$\boxed{PA = LU} \tag{38}$$

where P is a permutation matrix, L is lower triangular and U is upper triangular. As we have seen in the example above, P is obtained from A by first rearranging its rows in such a way that the pivots are in the standard order (i.e. ordered left-to-right as we go from top-to-bottom). Then L and U are obtained by the usual Gaussian elimination we described in the previous lecture.

Remark. *Look up to where the word "before" is underlined. If we replace this word by "after", and go Gaussian elimination by first applying elimination matrices and the applying row exchanges, then the factorization you get is $A = LPU$, where P is a permutation matrix. This is a variant of the $PA = LU$ decomposition described above.*

Another feature of matrices is the operation of taking **transposes**. Specifically, if A is an $m \times n$ matrix, its transpose A^T is an $n \times m$ matrix obtained by switching the roles of rows and columns:

$$\text{if } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

So the first row of A becomes the first column of A^T etc. In terms of entries, the ij entry of A is the ji entry of A^T, for all i and j. Transposes follow certain rules:

$$\boxed{(A+B)^T = A^T + B^T} \qquad \boxed{(AB)^T = B^T A^T} \tag{39}$$

$$\boxed{(A^T)^T = A} \qquad \boxed{(A^{-1})^T = (A^T)^{-1}} \tag{40}$$

(the last property requires A to be square and non-singular, or we couldn't talk about its inverse).

Transposes are related to a lot of things, for example, the dot product of vectors. Compare:

$$\text{the dot product} \qquad \begin{bmatrix} a_1 \\ ... \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix} = a_1 b_1 + ... + a_n b_n$$

$$\text{the matrix multiplication} \qquad \begin{bmatrix} a_1 \\ ... \\ a_n \end{bmatrix}^T \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 & ... & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 + ... + a_n b_n \end{bmatrix}$$

In other words, given two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, the only entry of the 1×1 matrix $\boldsymbol{v}^T \boldsymbol{w}$ is the same number as the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$.

Another instance of transposes comes about when studying **symmetric** matrices, namely square matrices which are equal to their own transposes:

$$\boxed{S \text{ is called symmetric if } S = S^T} \tag{41}$$

For example, the following matrix is symmetric:

$$\begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 8 \\ -1 & 8 & 0 \end{bmatrix}$$

because it is unchanged upon reflecting it across its diagonal. All diagonal matrices:

$$\begin{bmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{bmatrix}$$

are symmetric. More generally, here's a general recipe to obtain symmetric matrices:

$$\boxed{\text{for any matrix } A, \text{ the matrix } S = A^T A \text{ is symmetric}} \tag{42}$$

which holds even if the matrix A is rectangular. To prove this, note that (using (39) and (40)):

$$S^T = (A^T A)^T = A^T (A^T)^T = A^T A = S$$

Finally, let us note an important reason why symmetric matrices are important from a computational viewpoint. The LU factorization of a symmetric matrix takes a particularly nice form:

$$\boxed{S = LDL^T} \tag{43}$$

where L is a lower triangular matrix with 1's on the diagonal, and D is a diagonal matrix. So in other words, symmetric matrices are precisely those whose LDU decomposition has the property that $U = L^T$.

Lecture 6 (*September 16*) *Reading: section 3.1*

A key idea of linear algebra is to think of a line/plane/space as a **vector space**. Specifically, this means that points of n–dimensional space:

$$\mathbb{R}^n = \Big\{\text{set of all } n\text{–tuples of real numbers}\Big\} = \Big\{(x_1, ..., x_n),\ x_1, ..., x_n \text{ are real numbers}\Big\} \tag{44}$$

can be thought of as column vectors, according to the correspondence:

$$(x_1, ..., x_n) \quad \leftrightarrow \quad \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix}$$

For example, $\mathbb{R}$ is the line, $\mathbb{R}^2$ is the plane, $\mathbb{R}^3$ is usual space. But the key thing about thinking of these as sets of vectors is that vectors can be added, and vectors can be multiplied with scalars:

$$\begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ ... \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ ... \\ x_n + y_n \end{bmatrix} \qquad \text{and} \qquad \alpha \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ ... \\ \alpha x_n \end{bmatrix} \tag{45}$$

Definition 3. *A **vector space** is a set V, together with a notion of how to add two elements in V, and a notion of how to multiply an element of V by a scalar. In other words:*

$$\text{for any } \boldsymbol{v}, \boldsymbol{w} \in V, \text{ there must be a rule for defining } \boldsymbol{v} + \boldsymbol{w} \in V \tag{46}$$

$$\text{for any } \boldsymbol{v} \in V, \alpha \in \mathbb{R}, \text{there must be a rule for defining } \alpha \boldsymbol{v} \in V \tag{47}$$

This may seem abstract, but I will now spell out the main examples of vector spaces we will encounter in this course. First of all, $\mathbb{R}^n$ is a vector space, because of (45): the sum of vectors is a well-defined vector, and a vector times a scalar is a well-defined vector. More generally, we have:

Definition 4. *If V is a vector space, then a subset $S \subset V$ is called a **subspace** if:*

$$\text{for any } \boldsymbol{v}, \boldsymbol{w} \in S, \text{ then } \boldsymbol{v} + \boldsymbol{w} \in S \tag{48}$$

$$\text{for any } \boldsymbol{v} \in V, \alpha \in \mathbb{R}, \text{then } \alpha \boldsymbol{v} \in S \tag{49}$$

where the notions "$\boldsymbol{v} + \boldsymbol{w}$" and "$\alpha \boldsymbol{v}$" are well-defined, because S is a subset of a vector space V.

By combining (48) and (49), we have the more general fact for any subspace S of a vector space:

$$\text{for all } \boldsymbol{v}_1, ..., \boldsymbol{v}_k \in S \text{ and } \alpha_1, ..., \alpha_k \in \mathbb{R}$$

then:

$$\boxed{\alpha_1 \boldsymbol{v}_1 + ... + \alpha_k \boldsymbol{v}_k} \tag{50}$$

lies in S. For any scalars $\alpha_1, ..., \alpha_k$, the vector in (50) will be called a **linear combination** of the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$.

In this course, we will be interested in subspaces $S \subset \mathbb{R}^n$. There is a mind-boggling infinity of subsets of $\mathbb{R}^n$, but only a select few of them will turn out to be subspaces. In the case $n = 1$, there are only two subsets $S \subset \mathbb{R}$ which are subspaces: $S = \{0\}$ (the zero subspace) and $S = \mathbb{R}$ itself (the whole line). In the case $n = 2$, the subsets $S \subset \mathbb{R}^2$ which are subspaces are of three types:

- $S = \{0\}$ (the zero subspace)
- $S =$ any line passing through the origin
- $S = \mathbb{R}^2$ (the whole plane)

Indeed, if S contains any non-zero vector, then it contains the whole line passing though the origin and that vector, due to (49). And if S contains two non-zero vectors not on the same line, then it contains the entire plane, due to (50). Based on this pattern, you can probably guess that the subsets $S \subset \mathbb{R}^3$ which are subspaces are: the zero subspace, any line passing through the origin, any plane passing through the origin, and the whole space.

Now comes the point where we explain what all of this has to do with linear algebra, specifically with solving systems of equations like $A\boldsymbol{v} = \boldsymbol{b}$. We start from the following definition.

Definition 5. *The **column space** of an $m \times n$ matrix A is the subspace:*

$$C(A) \subset \mathbb{R}^m \tag{51}$$

spanned by the columns of A (the matrix A has n columns).

*Here and throughout, the subspace **spanned** by any vectors $\boldsymbol{v}_1$,...,$\boldsymbol{v}_k$ is defined to be the set of all linear combinations* (50) *of those vectors.*

Exercise: prove that the subspace spanned by any number of vectors in $\mathbb{R}^m$ is a subspace, in the sense of Definition 4. In the meantime, here is an example:

$$\text{if } A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \text{ then } C(A) = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \text{ for all scalars } \alpha \text{ and } \beta \right\}$$

So in other words, the column space in the example above is the plane traced out by the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\4\\6\end{bmatrix}$ in space. However:

$$\text{if } B = \begin{bmatrix}1 & 2\\2 & 4\\3 & 6\end{bmatrix} \text{ then } C(B) = \left\{ \alpha \begin{bmatrix}1\\2\\3\end{bmatrix} + \beta \begin{bmatrix}2\\4\\6\end{bmatrix}, \text{ for all scalars } \alpha \text{ and } \beta \right\} =$$

$$= \left\{ \gamma \begin{bmatrix}1\\2\\3\end{bmatrix}, \text{ for all scalars } \gamma \right\} \tag{52}$$

so the column space of B is the line determined by the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$. The reason for this is that the other column, namely $\begin{bmatrix}2\\4\\6\end{bmatrix}$, is already on the said line. Taking linear combinations of vectors that lie on the same line passing through the origin will still remain on the given line, because:

$$\alpha \begin{bmatrix}1\\2\\3\end{bmatrix} + \beta \begin{bmatrix}2\\4\\6\end{bmatrix} = \alpha \begin{bmatrix}1\\2\\3\end{bmatrix} + 2\beta \begin{bmatrix}1\\2\\3\end{bmatrix} = (\alpha + 2\beta) \begin{bmatrix}1\\2\\3\end{bmatrix}$$

for any scalars α and β.

Fact 5. *A system of equations $A\boldsymbol{v} = \boldsymbol{b}$ has solutions if and only if $\boldsymbol{b} \in C(A)$.*

The fact above is simply a restatement of Fact 2. It is simply saying that if there is a solution $\boldsymbol{v}$ to the system $A\boldsymbol{v} = \boldsymbol{b}$, then (5) implies that $\boldsymbol{b}$ is a linear combination of the columns of A. So the column space of a matrix precisely consists of those vectors $\boldsymbol{b}$ which can appear as the right-hand sides of systems of linear equations $A\boldsymbol{v} = \boldsymbol{b}$.

For example, does the system:

$$\begin{bmatrix}1 & 2\\2 & 4\\3 & 6\end{bmatrix} \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \tag{53}$$

have any solutions? The answer is no, because we have already seen in (52) that the column space of the matrix in question is the line spanned by the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$. Since the vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not on this line (as it is not a multiple of $\begin{bmatrix}1\\2\\3\end{bmatrix}$), then Fact 5 implies that the system (53) has no solutions.

Lecture 7 (*September 18*) *Reading: section 3.2*

Consider the subspace $S \subset \mathbb{R}^3$ spanned by the vectors:

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} \tag{54}$$

These two vectors are not on the same line (passing through the origin) so we conclude that S is a plane. As such, S is cut out by a single equation in the plane, and it is not hard to see what this equation is:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x + y + z = 0 \right\} \tag{55}$$

Indeed, both the vectors in (54) have the property that the sum of their entries is 0. Moreover, any linear combination of those two vectors has the property that the sum of its entries is 0, and this allows us to conclude (55). A more linear algebra way to say this is to consider the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

(the entries of A are precisely the coefficients of x, y and z in the equation of the plane (55)) and to observe the fact that:

$$A \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \qquad \text{and} \qquad A \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Moreover, taking appropriate linear combinations of these two equations allows us to conclude that:

$$A \left(\alpha \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 0 \end{bmatrix}$$

for any scalars α and β. Therefore, the plane S is precisely what people call the nullspace of A, as in the following definition.

Definition 6. *The **nullspace** of an $m \times n$ matrix A is the subspace:*

$$N(A) \subset \mathbb{R}^n \tag{56}$$

consisting of those vectors $\boldsymbol{v} \in \mathbb{R}^n$ such that $A\boldsymbol{v} = 0$.

Remark. *A little quirk concerning our notation: the right-hand side of the equation $A\boldsymbol{v} = 0$ should be an $m \times 1$ column vector, specifically the zero vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$. People usually abbreviate the zero vector by using the notation "0", although it is still technically a vector.*

The fact that the nullspace satisfies the requirements (48) and (49) of a subspace follows from the easy to prove relations:

$$A\boldsymbol{v} = 0 \text{ and } A\boldsymbol{w} = 0 \;\Rightarrow\; A(\boldsymbol{v}+\boldsymbol{w}) = 0$$

$$A\boldsymbol{v} = 0 \text{ and } \alpha \in \mathbb{R} \quad \Rightarrow \quad A(\alpha\boldsymbol{v}) = 0$$

So computing the nullspace of a matrix A boils down to solving the equation $A\boldsymbol{v} = 0$. However, many matrices have the same nullspace. For example, if:

$$\boxed{A = LR} \tag{57}$$

where L is an invertible matrix (such as the product of elimination/permutation/diagonal matrices, hint-hint) then:

$$\boxed{N(A) = N(R)} \tag{58}$$

which is essentially saying that:

$$A\boldsymbol{v} = LR\boldsymbol{v} = 0 \qquad \text{is equivalent to} \qquad R\boldsymbol{v} = 0$$

Gauss-Jordan elimination gives us a recipe for writing A in the form (57), where R is in reduced row echelon form, i.e.:

- the pivots are all 1
- the pivots go to the right as we read the rows from top to bottom
- all the entries above a pivot are zeroes

For example, the following matrix is in reduced row echelon form:

$$R = \begin{bmatrix} \boxed{1} & 3 & 0 & 7 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

(as usual, the pivots are in boxes). A column which has a pivot is called a (unsurprisingly) **pivot column** and all other columns are called **free columns**. In the example of R above, the first and third are pivot columns, and the second and fourth are free columns. Explicitly, we have:

$$R\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \qquad \text{means that} \qquad \begin{bmatrix} a+3b+7d \\ c+2d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{i.e.} \qquad \begin{cases} a = -3b-7d \\ c = -2d \end{cases}$$

So here's the catch: for any choice of b and d, the equalities above tell you what a and c have to be in order for the vector:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

to lie in the nullspace of R. Thus, b and d are **free variables**, and they precisely correspond to the free columns of R. We conclude that the nullspace of R can be explicitly described as:

$$N(R) = \left\{ \begin{bmatrix} -3b - 7d \\ b \\ -2d \\ d \end{bmatrix} \text{ for all } b, d \text{ real numbers} \right\}$$

The set above is simply a plane in 4-dimensional space. If you want to describe this plane as being generated by two vectors, then all you need to do is to assign values to the free variables. A reasonable choice is to set $(b, d) = (1, 0)$ or $(0, 1)$, and this implies that:

$$N(R) \text{ is spanned by the vectors } \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -7 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, in general, here is the algorithm for computing the nullspace of any $m \times n$ matrix A (or equivalently, for finding all solutions to the system of equations $A\boldsymbol{v} = 0$):

- Write A in reduced row echelon form, and call the resulting matrix R

- Identify the pivot columns and the free columns of R; the pivot/free variables will be those x_i among $x_1, ..., x_n$ such that the i–th column of R is a pivot/free column

- The nullspace of A, which coincides with the nullspace of R, is spanned by those vectors $(x_1, ..., x_n)$ where a given free variable is 1, all other free variables are 0, and all the pivot variables are determined by the free variables and the equations:

$$R \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

  It is easy to solve for the pivot variables (in terms of the free variables) in the equation above by back-substitution, precisely because R is in reduced row echelon form.

Lecture 8 (*September 23*) *Reading: section 3.3*

We will now put together the ideas of the previous two lectures in order to find the complete (a.k.a. general) solution $\boldsymbol{v}$ of the system:

$$A\boldsymbol{v} = \boldsymbol{b} \tag{59}$$

for any $m \times n$ matrix A. First of all, as we have seen in Fact 5, there are no solutions if the $m \times 1$ vector $\boldsymbol{b}$ is not in the column space $C(A)$. So we henceforth assume that $\boldsymbol{b} \in C(A)$, which is just saying that there exists a particular solution:

$$A\boldsymbol{v}_{\text{particular}} = \boldsymbol{b} \tag{60}$$

This $n \times 1$ vector $\boldsymbol{v}_{\text{particular}}$ is a solution of (59), but it may not be the only one. In fact, for any other solution $\boldsymbol{v}$ of the original system (59), we may subtract the two equations above and obtain:

$$A(\boldsymbol{v} - \boldsymbol{v}_{\text{particular}}) = 0$$

So the difference $\boldsymbol{v} - \boldsymbol{v}_{\text{particular}}$ lies in the nullspace of A. This immediately leads to the following.

Fact 6. *The general solution of the system* (59) *is given by:*

$$\boxed{\boldsymbol{v}_{\text{general}} = \boldsymbol{v}_{\text{particular}} + \boldsymbol{w}_{\text{general}}} \tag{61}$$

where:

- $\boldsymbol{v}_{\text{particular}}$ *is a particular solution of* (59)
- $\boldsymbol{w}_{\text{general}}$ *is any element of* $N(A)$*, i.e. the general solution to* $A\boldsymbol{w}_{\text{general}} = 0$

The fact above is a general mathematical feature of linear equations. For example, it also holds in the theory of linear ordinary differential equations, as you can see in 18.03.

Reduced row echelon forms allow you to deal with both bullets in Fact 6 at the same time. Let me show you how to do this in an example. Consider the system:

$$\begin{bmatrix} 2 & 6 & 2 & 2 & -1 \\ 2 & 6 & 1 & -1 & 2 \\ 3 & 9 & -1 & -9 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \\ -6 \end{bmatrix} \tag{62}$$

Let us now apply Gauss-Jordan elimination to the augmented matrix of the system:

$$\left[\begin{array}{ccccc|c} 2 & 6 & 2 & 2 & -1 & -7 \\ 2 & 6 & 1 & -1 & 2 & 6 \\ 3 & 9 & -1 & -9 & 1 & -6 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccccc|c} \boxed{1} & 3 & 0 & -2 & 0 & -3 \\ 0 & 0 & \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 5 \end{array}\right]$$

The matrix on the right is in reduced row echelon form (the pivots are boxed) and it corresponds to the system:

$$\begin{bmatrix} \boxed{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} \tag{63}$$

By Fact 3, finding the general solution of the system (62) is equivalent to finding the solution of the system (63), so let us solve the latter. To do so, let us look at the coefficient matrix:

$$R = \begin{bmatrix} \boxed{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

and identify the pivot columns (these are columns 1,3,5 in the example at hand) and the free columns (these are columns 2,4 in the example at hand). Therefore, the pivot variables are x_1, x_3, x_5 and the free variables are x_2, x_4. Now we make the following observation:

For any choice of free variables x_2, x_4 there exists a unique choice of pivot variables x_1, x_3, x_5 which solve the system (63)

So we keep x_2, x_4 as arbitrary numbers, and solve for x_1, x_3, x_5 in (63) by back substitution:

$$\begin{cases} x_1 + 3x_2 - 2x_4 = -3 \\ x_3 + 3x_4 = 2 \\ x_5 = 5 \end{cases} \quad \rightsquigarrow \quad \begin{cases} x_1 = -3 - 3x_2 + 2x_4 \\ x_3 = 2 - 3x_4 \\ x_5 = 5 \end{cases}$$

So we conclude that the general solution to the system (63) (which is the same as the general solution to the system (62), by the very nature of Gauss-Jordan elimination) is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{\text{general}} = \begin{bmatrix} -3 - 3a + 2b \\ a \\ 2 - 3b \\ b \\ 5 \end{bmatrix} \tag{64}$$

(we relabeled $x_2 = a$ and $x_4 = b$ to emphasize the fact that they are arbitrary real numbers). So what if you didn't need the general solution of (62) and you were satisfied with a particular solution? One easy way to get a particular solution is to just plug in $a = b = 0$, thus obtaining:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{\text{particular}} = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 5 \end{bmatrix} \tag{65}$$

You could also have gotten this by setting all the free variables equal to 0 in (63) and solving for the pivot variables by using back substitution. Similarly, if you subtract the particular solution (65) from the general solution (64), you get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{\text{general}} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{\text{particular}} = \begin{bmatrix} -3a + 2b \\ a \\ -3b \\ b \\ 0 \end{bmatrix} \tag{66}$$

The right-hand side is just the general element of the nullspace of the matrix R.

Let us just mention one thing which could have gone wrong in the procedure above, and how to remedy it. It could have been that the reduced row echelon form of the coefficient matrix had a full row of zeroes. For example, instead of the system (63) we could have had:

$$\begin{bmatrix} \boxed{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} \tag{67}$$

(when doing elimination, it is customary to put all the zero rows on the very bottom of the matrix, which we can achieve by suitable row exchanges). It is easy to see that the system above has no solutions, because equating the third rows of equality (67) to each other would force us to have $0 = 5$, which is impossible. However, if the system were instead:

$$\begin{bmatrix} \boxed{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} \tag{68}$$

then we do have solutions. To find these solutions, just throw out the zero row at the bottom of the equality (68) and proceed as we did on the previous page. The upshot is, for any matrix R in reduced row echelon form:

a system $R\boldsymbol{v} = \boldsymbol{c}$ has solutions if and only if the vector

$\boldsymbol{c}$ has entry 0 corresponding to every full row of zeroes of R

Definition 7. *The rank r of a matrix A is equal to the number of its pivot columns.*

Here and throughout, the pivot columns of a matrix A refer to those columns where the pivots lie in the (reduced) row echelon form of A.

If the matrix A is $m \times n$, then the rank satifies the inequality:

$$\boxed{r \leq \min(m, n)}$$

since the number of pivots cannot surpass either the number of rows or of columns. We say that:

- A has full row rank if $r = m$. In this case, the column space of A is as big as possible, i.e. $C(A) = \mathbb{R}^m$, and the system $A\boldsymbol{v} = \boldsymbol{b}$ have at least one solution $\boldsymbol{v}$ for any $\boldsymbol{b}$.

- A has full column rank if $r = n$. In this case, the nullspace of A is as small as possible, i.e. $N(A) = 0$, and the system $A\boldsymbol{v} = \boldsymbol{b}$ have at most one solution $\boldsymbol{v}$ for any $\boldsymbol{b}$.

Lecture 9 (*September 25*) *Reading: section 3.4*

Given vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n \in \mathbb{R}^m$, recall that the subspace V spanned by these vectors was explained in Definition 5. Specifically, vectors in the subspace $V \subset \mathbb{R}^m$ are arbitrary linear combinations:

$$\alpha_1 \boldsymbol{v}_1 + ... + \alpha_n \boldsymbol{v}_n \tag{69}$$

for various scalars $\alpha_1, ..., \alpha_n$. But what if one of the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ is itself a linear combination of the other vectors? For example, we could have:

$$\boldsymbol{v}_i = \beta_1 \boldsymbol{v}_1 + ... + \beta_{i-1} \boldsymbol{v}_{i-1} + \beta_{i+1} \boldsymbol{v}_{i+1} + ... + \beta_n \boldsymbol{v}_n$$

for some $i \in \{1, ..., n\}$. Then any linear combination of the form (69) can be written as a linear combination of the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, ..., \boldsymbol{v}_n$ only:

$$\alpha_1\boldsymbol{v}_1 + ... + \alpha_{i-1}\boldsymbol{v}_{i-1} + \alpha_i \underbrace{(\beta_1\boldsymbol{v}_1 + ... + \beta_{i-1}\boldsymbol{v}_{i-1} + \beta_{i+1}\boldsymbol{v}_{i+1} + ... + \beta_n\boldsymbol{v}_n)}_{\boldsymbol{v}_i} + \alpha_{i+1}\boldsymbol{v}_{i+1} + ... + \alpha_n\boldsymbol{v}_n =$$

$$= (\alpha_1 + \alpha_i\beta_1)\boldsymbol{v}_1 + ... + (\alpha_{i-1} + \alpha_i\beta_{i-1})\boldsymbol{v}_{i-1} + (\alpha_{i+1} + \alpha_i\beta_{i+1})\boldsymbol{v}_{i+1} + ... + (\alpha_n + \alpha_i\beta_n)\boldsymbol{v}_{i+1}$$

So we don't need the vector $\boldsymbol{v}_i$ to span the subspace $V \subset \mathbb{R}^m$: this subspace is in fact spanned by the slightly smaller collection of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, ..., \boldsymbol{v}_n$.

Definition 8. *Vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n \in \mathbb{R}^m$ are called **(linearly) independent** if none of them is a linear combination of the others. Equivalently, this means that any linear combination is non-zero:*

$$\alpha_1\boldsymbol{v}_1 + ... + \alpha_n\boldsymbol{v}_n \neq 0 \tag{70}$$

except for the zero combination (namely the one where $\alpha_1 = ... = \alpha_n = 0$).

If the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n \in \mathbb{R}^m$ are not independent, we call them **(linearly) dependent**, which means that there exists a linear combination:

$$\alpha_1\boldsymbol{v}_1 + ... + \alpha_n\boldsymbol{v}_n = 0 \tag{71}$$

where not all the $\alpha_1, ..., \alpha_n$ are 0. For example, if $\alpha_i \neq 0$, then we can rewrite equation (71) as:

$$\boldsymbol{v}_i = -\frac{\alpha_1}{\alpha_i}\boldsymbol{v}_1 - ... - \frac{\alpha_{i-1}}{\alpha_i}\boldsymbol{v}_{i-1} - \frac{\alpha_{i+1}}{\alpha_i}\boldsymbol{v}_{i+1} - ... - \frac{\alpha_n}{\alpha_i}\boldsymbol{v}_n$$

We conclude that a collection of vectors is linear dependent precisely when one of them can be written as a linear combination of the others.

For example, the two vectors:

$$\begin{bmatrix}2\\5\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\-3\\-1\end{bmatrix} \tag{72}$$

are linearly independent, because otherwise one of them would be a linear combination (i.e. a multiple) of the other, and this is visibly not true. Geometrically, we can see this because the two vectors above are not on the same line in space. However, the three vectors:

$$\begin{bmatrix}2\\5\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\-3\\-1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}4\\-1\\-1\end{bmatrix} \tag{73}$$

are linearly dependent, because the following linear combination is 0:

$$1 \cdot \begin{bmatrix}2\\5\\1\end{bmatrix} + 2 \cdot \begin{bmatrix}1\\-3\\-1\end{bmatrix} - 1 \cdot \begin{bmatrix}4\\-1\\-1\end{bmatrix} = 0 \tag{74}$$

Definition 9. *A **basis** of a vector space V is a collection of independent vectors which span V.*

For example, both the collection (72) and the collection (73) span the plane:

$$\{2x - 3y + 11z = 0\} \subset \mathbb{R}^3 \tag{75}$$

But only the collection of vectors (72) is a basis, because they are linearly independent. The collection of vectors (73) is not a basis, because each of the three vectors in (73) can be written as a linear combination of the other two (see (74)). So any of these three vectors is redundant in spanning the plane (75), and that is why we don't need all three of them to have a basis.

Remark. *A basis of any vector space is not unique. In fact, any collection of two of the three vectors in* (73) *form a basis of the subspace* (75)*.*

Definition 10. *The* ***dimension*** *of a vector space V is the number of vectors in a basis of V.*

Implicit in the Definition above is that all bases of a vector space (and there are infinitely many of them, see the Remark above) have the same number of vectors in them. The abstract notion of dimension introduced above precisely matches our usual, geometric, notion of dimension: a line has dimension 1 (it is spanned by a single vector) and a plane has dimension 2 (it is spanned by two vectors, for example the plane (75) with the basis (72)).

So, practically, how can we tell when a collection of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ is linearly independent? And if they are not linearly independent, how can we remove some of these vectors, in such a way as to obtain a basis of the subspace they span? The way to answer both of these questions is to construct the $m \times n$ matrix whose columns are the vectors in question:

$$A = \left[\begin{array}{c|c|c} \boldsymbol{v}_1 & \dots & \boldsymbol{v}_n \end{array} \right]$$

For any vector $\boldsymbol{w} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$, we have seen that:

$$A\boldsymbol{w} = \alpha_1 \boldsymbol{v}_1 + ... + \alpha_n \boldsymbol{v}_n \tag{76}$$

Therefore, the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are linearly independent if and only if $A\boldsymbol{w} = 0$ happens only when $\boldsymbol{w} = 0$, or in other words, if $N(A) = 0$. Conversely, the vectors are dependent precisely when there exists $\boldsymbol{w} \neq 0$ such that $A\boldsymbol{w} = 0$. So to find a linear combination (76) which is 0, one only needs to find a non-zero vector $\boldsymbol{w} \in N(A)$, and this can be done by putting A in reduced row echelon form.

If the vectors $\boldsymbol{v}_1$,..., $\boldsymbol{v}_n$ are linearly dependent, how to we find a subset of them which are linearly independent, and thus give a basis of the vector subspace they span? Simple: put A in reduced row echelon form, identify the pivot columns, and look at the corresponding columns of A. Those will be a basis. For example, consider the matrix:

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 5 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} R = \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are columns 1 and 2, so the upshot is that a basis of $C(A)$ consists of the first two columns of A. This is precisely the basis we found in (72).

Fact 7. *The rank of the matrix A (i.e. the number of pivot columns) equals the dimension of $C(A)$.*

Above, we saw how to find a basis of the vector space spanned by a collection of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$. But there are other ways to present vector spaces, namely by equations. For example consider:

$$V = \{2x - 2y - 6z - 4t = 5x + y - 3z + 8t = -3x + 2y + 7z + 3t = 0\} \subset \mathbb{R}^4$$

Since V is cut out by three equations in four unknowns, it is tempting to believe that V is one-dimensional. But not so fast! Let's find a basis of V systematically. The way to do so is to think of V as the nullspace of the following matrix:

$$V = N(A) \quad \text{where} \quad A = \begin{bmatrix} 2 & -2 & -6 & -4 \\ 5 & 1 & -3 & 8 \\ -3 & 2 & 7 & 3 \end{bmatrix}$$

To describe this nullspace, put the matrix A in reduced row echelon form:

$$A = \begin{bmatrix} 2 & -2 & -6 & -4 \\ 5 & 1 & -3 & 8 \\ -3 & 2 & 7 & 3 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} R = \begin{bmatrix} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $N(A) = N(R)$, the nullspace of the matrix R in reduced row echelon form can be described in terms of pivot variables (x and y) and free variables (z and t):

$$R \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0 \quad \text{i.e.} \quad \begin{cases} x - z + t = 0 \\ y + 2z + 3t = 0 \end{cases}$$

Hence:

$$V = \left\{ \begin{bmatrix} z - t \\ -2z - 3t \\ z \\ t \end{bmatrix} \text{ for any } z, t \right\}$$

To find basis vectors of V, just set one of the free variables (in this case, either z or t) equal to 1 and the other one equal to 0:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, V is two-dimensional, hence a plane.

Let us summarize the two ways of thinking about a vector $V \subset \mathbb{R}^m$: either as the column space of a matrix (when you are given a set of vectors which spans V), or as the nullspace of a matrix (when you are given equations satisfied by V). In either case, analyzing V (i.e. finding a basis and computing its dimension) is best done by putting the matrix in reduced row echelon form.

Remark. *If A is a square $n \times n$ matrix, the rank always satisfies $r \leq n$. We can only have equality, i.e. $r = n$, if the columns of A are linearly independent. This is equivalent to the only solution of $A\boldsymbol{v} = 0$ being the zero vector $\boldsymbol{v} = 0$, so the nullspace of A just consists of the zero vector. Moreover:*

$$\boxed{r = n} \quad \textit{if and only if} \quad \boxed{A \textit{ is invertible}} \tag{77}$$

Indeed, everything we said in this Remark can be translated as saying that the system $A\boldsymbol{v} = \boldsymbol{b}$ has a single solution $\boldsymbol{v}$ for any vector $\boldsymbol{b}$. This solution si precisely $\boldsymbol{v} = A^{-1}\boldsymbol{b}$.

Lecture 10 (*September 27*) *Reading: section 3.5*

To a $m \times n$ matrix A, we associated the following subspaces:

$$\text{the column space } C(A) \subset \mathbb{R}^m$$
$$\text{the nullspace } N(A) \subset \mathbb{R}^n$$

We will now consider two more subspaces corresponding to the matrix A:

$$\text{the row space } C(A^T) \subset \mathbb{R}^n$$
$$\text{the left nullspace } N(A^T) \subset \mathbb{R}^m$$

Collectively, the four subspaces above are called the **four fundamental subspaces** of A. The row space is just the subspace spanned by the rows of the matrix A (which are the same things as the columns of the transpose matrix A^T). Meanwhile, the left nullspace consists of $m \times 1$ vectors such that:

$$A^T \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = 0$$

If you transpose the equation above, then you obtain:

$$\begin{bmatrix} v_1 & \dots & v_m \end{bmatrix} A = 0$$

This leads to the following description of the left nullspace of a matrix A: it consists of $1 \times m$ vectors (in other words, row vectors) $\boldsymbol{v}$ such that:

$$\boxed{\boldsymbol{v}A = 0} \tag{78}$$

This equation explains why we call it the "left" nullspace.

Fact 8. *If r is the rank of the matrix A (number of pivots), then the four subspaces have dimensions:*

$$\boxed{\dim C(A) = r} \qquad \boxed{\dim C(A^T) = r}$$

$$\boxed{\dim N(A) = n - r} \qquad \boxed{\dim N(A^T) = m - r}$$

Let's try and justify the fact above. The fact that the column space has dimension equal to r is just a consequence of the fact that the pivot columns form a basis of $C(A)$. The fact that the nullspace has dimension $n - r$ follows from the fact a choice of basis vectors of $N(A)$ is given by setting all the free variables equal to 0 except for one of them; the number of such basis vectors is the same as the number of free variables, namely n minus the number of pivot variables, i.e. $n-r$.

Let's justify the fact that the row space of A also has dimension r. To see this, let us bring the matrix in reduced row echelon form R (row operations do not change the row space). Of the m rows of R, exactly r of them will be non-zero (the rows which have pivots) and $m - r$ of them will be zero (the rows which do not have pivots). Therefore, the dimension of the row space is exactly r, since pivot rows are linearly independent. This implies that $\dim C(A^T) = r$, which also implies that $\dim N(A^T) = m-r$, just by repeating the argument in the previous paragraph for A replaced by A^T.

The reason why we introduced the row space and the left nullspace is that they provide complements for the nullspace and column space, respectively. To understand what this means, suppose you had two vector subspaces:

$$V, W \subset \mathbb{R}^n$$

of dimensions k and $n - k$, respectively. If V and W are sufficiently general, then their intersection only consists of the zero vector, in which case we call V and W **complementary** subspaces. This notion only applies to when the dimensions of V and W add up to n, the dimension of ambient space.

Example 2. *Two different lines passing through the origin are complementary subspaces of* $\mathbb{R}^2$*, and any plane and line which only intersect at the origin are complementary subspaces of* $\mathbb{R}^3$*.*

However, two planes passing through the origin can never be complementary subspaces in $\mathbb{R}^3$*, because their intersection always contains a line (moreover, the dimensions add up to* $2+2 = 4 \neq 3$*).*

Here is a fact we will understand more closely in upcoming weeks:

The row space and nullspace of A are complementary subspaces of $\mathbb{R}^n$

The column space and left nullspace of A are complementary subspaces of $\mathbb{R}^m$

for any $m \times n$ matrix A. But practically, how do we compute the four fundamental subspaces of a matrix? For example, how do we find a basis of each of them? For the column space and nullspace of A, we saw that the key role was played by its reduced row echelon form R:

- the column space $C(A)$ is spanned by the pivot columns of A (i.e. those columns where the reduced row echelon form R has its pivots)
- the nullspace $N(A)$ is spanned by the vectors $\boldsymbol{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ where all the free variables except for one take the value 0, and the pivot variables take values prescribed by $R\boldsymbol{v} = 0$

So how do we compute the spaces $C(A^T)$ and $N(A^T)$? The direct way is to just put A^T in reduced row echelon form, and apply the previous two bullets. But quite often, the problem naturally deals with the reduced row echelon form of the matrix A itself:

$$A \overset{\text{RREF}}{\rightsquigarrow} R \tag{79}$$

By analogy with (58), we have:

$$\boxed{C(A^T) = C(R^T)} \tag{80}$$

which happens because row operations do not change the row space of a matrix. Therefore, a basis for $C(A^T)$ is given by the pivot rows of the matrix R.

What about the left nullspace $N(A^T)$ in terms of the reduced row echelon form (79)? As we have seen a few lectures ago, Gauss-Jordan elimination involves multiplying the matrix A on the left:

$$KA = R$$

by an invertible matrix K. By (78), a row vector $\boldsymbol{v}$ lies in $N(A^T)$ precisely means:

$$0 = \boldsymbol{v}A = \boldsymbol{v}K^{-1}R \tag{81}$$

where $\boldsymbol{v}$ is an $1 \times m$ vector. The reduced row echelon matrix R will have the form, say:

$$\begin{bmatrix} \boxed{1} & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & \boxed{1} & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $0 = \boldsymbol{w}R$ happens if and only if:

$$\boldsymbol{w} = \begin{bmatrix} 0 & 0 & 0 & * & * \end{bmatrix}$$

(namely the only non-zero entries of $\boldsymbol{w}$ are those corresponding to the zero rows of R). So the vectors $\boldsymbol{v}$ that can satisfy (81) are those of the form:

$$\boldsymbol{v} = \boldsymbol{w}K = \begin{bmatrix} 0 & 0 & 0 & * & * \end{bmatrix} K$$

We conclude that a basis for $N(A^T)$ is given by the rows of the square matrix K which correspond to the zero rows of the reduced row echelon form matrix R.

Lecture 11 (*September 30*) *Reading: section 4.1*

Today we will start discussing orthogonality (i.e.perpendicularity). First of all, we know that two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ are orthogonal precisely when their dot product is 0:

$$\boldsymbol{v} \perp \boldsymbol{w} \qquad \Leftrightarrow \qquad \boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T\boldsymbol{w} = 0 \tag{82}$$

If two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal, then we have the Pythagorean theorem:

$$||\boldsymbol{v} + \boldsymbol{w}||^2 = ||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2$$

(proof: $||\boldsymbol{v}+\boldsymbol{w}||^2 = (\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w} = ||\boldsymbol{v}||^2+||\boldsymbol{w}||^2$).

But now we have the necessary language to talk about more than just individual vectors in $\mathbb{R}^n$, and so we can generalize the concept of orthogonality to subspaces:

Definition 11. *Two subspaces $V, W \subset \mathbb{R}^n$ are called **orthogonal**, which will be denoted by $V \perp W$, if we have $\boldsymbol{v} \perp \boldsymbol{w}$ for any two vectors $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$.*

Note that two subspaces can be orthogonal only if $\dim V + \dim W \leq n$. Otherwise, the two subspaces would intersect in a non-zero vector $\boldsymbol{v} \in V \cap W$ (think about two planes passing through the origin in $\mathbb{R}^3$: they always contain a whole line) and it would be impossible for $\boldsymbol{v} \perp \boldsymbol{v}$.

Let us now consider an $m \times n$ matrix A, with the four fundamental subspaces considered in the previous Subsection. We have the following facts:

$$\boxed{C(A) \perp N(A^T)} \tag{83}$$

$$\boxed{N(A) \perp C(A^T)} \tag{84}$$

Let's prove these formulas: for (83), we need to take an arbitrary vector $\boldsymbol{a} \in C(A)$ and an arbitrary vector $\boldsymbol{v} \in N(A^T)$ and show that they are perpendicular. By the very definition of the column space and the left nullspace, we have:

$$\boldsymbol{a} = A\boldsymbol{w} \text{ for some vector } \boldsymbol{w} \qquad \text{and} \qquad A^T\boldsymbol{v} = 0 \;\Leftrightarrow\; \boldsymbol{v}^T A = 0$$

Then we have:

$$\boldsymbol{v}\cdot\boldsymbol{a} = \boldsymbol{v}^T\boldsymbol{a} = \boldsymbol{v}^T A\boldsymbol{w} = 0\boldsymbol{w} = 0 \qquad \Rightarrow \qquad \boxed{\boldsymbol{a} \perp \boldsymbol{v}}$$

Similarly, let's prove (84): we need to take an arbitrary $\boldsymbol{w} \in N(A)$ and an arbitrary $\boldsymbol{b} \in C(A^T)$ and show that they are perpendicular. By the very definition of nullspace and row space, we have:

$$A\boldsymbol{w} = 0 \qquad \text{and} \qquad \boldsymbol{b} = A^T\boldsymbol{v} \text{ for some vector } \boldsymbol{v}$$

Then we have:

$$\boldsymbol{b}^T = \boldsymbol{v}^T A \quad \Rightarrow \quad \boldsymbol{b}^T\boldsymbol{w} = \boldsymbol{v}^T A\boldsymbol{w} = \boldsymbol{v}^T 0 = 0 \quad \Rightarrow \quad \boxed{\boldsymbol{b} \perp \boldsymbol{w}}$$

(alternatively, you can think of statement (84) as simply restating (83) for A^T instead of A). Moreover, we saw in Fact 8 that the spaces in (83) are of complementary dimension, and thus they have the maximal total dimension that orthogonal subspaces can have. Ditto for (84). This is a special case of the following notion:

Definition 12. *Given a subspace $V \subset \mathbb{R}^n$, its **orthogonal complement** is the subspace $V^\perp$ consisting of all vectors in $\mathbb{R}^n$ perpendicular to V. In other words:*

$$V^\perp = \Big\{ \boldsymbol{w} \in \mathbb{R}^n \text{ such that } \boldsymbol{v} \perp \boldsymbol{w} \text{ for all } \boldsymbol{v} \in V \Big\} \tag{85}$$

Note that (85) is just a set. To show that it is a subspace, you would need to prove that:

$$\boldsymbol{v} \perp \boldsymbol{w}_1 \text{ and } \boldsymbol{v} \perp \boldsymbol{w}_2 \text{ for all } \boldsymbol{v} \in V \qquad \Rightarrow \qquad \boldsymbol{v} \perp (\alpha_1 \boldsymbol{w}_1 + \alpha_2 \boldsymbol{w}_2) \text{ for all } \boldsymbol{v} \in V$$

for any scalars α_1, α_2. The implication above can be easily proved by resorting to the dot product interpretation of perpendicularity. So combining Fact 8 with (83), (84) precisely says that:

the left nullspace is the orthogonal complement of the column space

the row space is the orthogonal complement of the nullspace

This is sometimes called the Fundamental Theorem of Linear Algebra. Let's see an example:

$$A = \begin{bmatrix} \boxed{1} & 0 & 3 & -1 \\ 0 & \boxed{1} & -2 & 1 \end{bmatrix}$$

(I chose a matrix which is already in reduced row echelon form to save us the work of Gauss-Jordan elimination). Then the row space of A is:

$$C(A^T) = \text{spanned by } \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \tag{86}$$

(we write the rows vertically to keep with our convention that vectors should be written as columns). To compute a basis for the nullspace of A, you must first note that the pivot columns of the matrix are 1 and 2, while the free columns are 3 and 4. Basis vectors for the nullspace are given by setting all the free variables equal to 0, except for one which is set equal to 1:

$$N(A) = \text{spanned by } \begin{bmatrix} a \\ b \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} c \\ d \\ 0 \\ 1 \end{bmatrix}$$

To solve for a, b, c, d, you must use the property that the vectors above are killed by A:

$$A \begin{bmatrix} a \\ b \\ 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} c \\ d \\ 0 \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{cases} a + 3 = 0 \\ b - 2 = 0 \\ c - 1 = 0 \\ d + 1 = 0 \end{cases}$$

so we conclude that:

$$N(A) = \text{spanned by } \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \tag{87}$$

It is easy to see that the two basis vectors in (86) are orthogonal to each of the two basis vectors in (87), thus establishing the fact that the row space and nullspace are orthogonal subspaces.

The fundamental theorem of linear algebra says that the four fundamental subspaces of a matrix are pairwise orthogonal complements. Here's why we like this setup: if you have two complemetary subspaces $V, W \subset \mathbb{R}^n$ (as defined in Lecture 10) then any vector $\boldsymbol{a} \in \mathbb{R}^n$ can be written as:

$$\boldsymbol{a} = \boldsymbol{v} + \boldsymbol{w} \quad \text{where } \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W \tag{88}$$

Moreover, $\boldsymbol{v}$ and $\boldsymbol{w}$ with this property are unique, and they are called the **components** of the vector $\boldsymbol{a}$ in the complementary subspaces V and W, respectively. So for example, letting V and W be the nullspace and row space of an $m \times n$ matrix, then we conclude that any vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of the rows of A plus a vector killed by A.

Lecture 12 (*October 4*) *Reading: section 4.2*

In the special case when V, W are complementary orthogonal subspaces of $\mathbb{R}^n$ (i.e. $V = W^\perp$, or equivalently, $W = V^\perp$), the decomposition (88) takes on a more geometric meaning. In this case, $\boldsymbol{v}$ and $\boldsymbol{w}$ are none other than the projections of $\boldsymbol{a}$ onto the subspaces V and W, as in the following:

Definition 13. *Given a vector $\boldsymbol{b} \in \mathbb{R}^n$ and a linear subspace $V \subset \mathbb{R}^n$, the **orthogonal projection** of $\boldsymbol{b}$ onto V is the unique vector:*

$$\boldsymbol{p} = \text{proj}_V \boldsymbol{b} \in V$$

such that $(\boldsymbol{b} - \boldsymbol{p}) \perp V$. Put differently, $\boldsymbol{p}$ is the closest vector in V to $\boldsymbol{b}$.

Intuitively, suppose you want to measure a quantity, which is represented by a vector $\boldsymbol{p}$ in a subspace $V \subset \mathbb{R}^n$ (for example, the collection of electric currents through n pieces of wire, which are constrained to live in a subspace of $\mathbb{R}^n$ by Kirchoff's law). However, your measurement will probably have an error, and you might get a value $\boldsymbol{b}$ outside of V. Then your best guess for the actual value $\boldsymbol{p} \in V$ is to set it equal to the orthogonal projection of $\boldsymbol{b}$ on V, since this is the closest vector in V to the measured value $\boldsymbol{b}$. That is also why the vector $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is called the **error**.

Let us work out formulas for the projection of a vector $\boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. The easiest case is when V is one dimensional, i.e. the line spanned by a vector $\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Then the orthogonal projection $\boldsymbol{p}$ must be a multiple of this vector, so we are looking for:

$$\boldsymbol{p} = \lambda \boldsymbol{a} \quad \text{such that} \quad (\boldsymbol{b} - \lambda \boldsymbol{a}) \perp \boldsymbol{a}$$

Using the dot product interpretation of perpendicularity, we need:

$$0 = (\boldsymbol{b} - \lambda \boldsymbol{a}) \cdot \boldsymbol{a} \quad \Rightarrow \quad \lambda = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{a}} = \frac{\boldsymbol{a}^T \boldsymbol{b}}{\boldsymbol{a}^T \boldsymbol{a}}$$

So the formula for the projection of $\boldsymbol{b}$ onto the line spanned by $\boldsymbol{a}$ is:

$$\text{proj}_{\boldsymbol{a}} \boldsymbol{b} = \boxed{\boldsymbol{a} \quad \underbrace{\frac{\boldsymbol{a}^T \boldsymbol{b}}{\boldsymbol{a}^T \boldsymbol{a}}}_{\text{this is a scalar}}} \tag{89}$$

Here's a sanity check: assume $\boldsymbol{a} = \boldsymbol{e}_i$ is the unit vector on the i–th axis. Then formula (89) reads:

$$\text{proj}_{\boldsymbol{e}_i} \boldsymbol{b} = \boldsymbol{e}_i b_i \tag{90}$$

so the projection only remembers the i–th component of the vector $\boldsymbol{b}$. It is easy to see that the operation (90) can be presented as a matrix multiplying the vector $\boldsymbol{b}$, specifically:

$$P_i \boldsymbol{b} = \boldsymbol{e}_i b_i \qquad \text{where} \quad P_i = \begin{bmatrix} ... & ... & 0 & ... & ... \\ ... & ... & 0 & ... & ... \\ 0 & 0 & 1 & 0 & 0 \\ ... & ... & 0 & ... & ... \\ ... & ... & 0 & ... & ... \end{bmatrix} \tag{91}$$

with the unique 1 being at the intersection of row i and column i. Any projection onto a line can be described by a similar matrix, specifically:

$$P_{\boldsymbol{a}} \boldsymbol{b} = \boldsymbol{a} \frac{\boldsymbol{a}^T \boldsymbol{b}}{\boldsymbol{a}^T \boldsymbol{a}} \qquad \text{where} \qquad P_{\boldsymbol{a}} = \frac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T \boldsymbol{a}} \tag{92}$$

Indeed, since $\boldsymbol{a}$ is an $n \times 1$ vector, the numerator of $P_{\boldsymbol{a}}$ is an $n \times n$ matrix, and the denominator is a 1×1 matrix, i.e. a scalar. This all fits into the following principle:

Fact 9. *For any subspace $V \subset \mathbb{R}^n$, there exists an $n \times n$ matrix P_V such that:*

$$\boxed{\text{proj}_V \boldsymbol{b} = P_V \boldsymbol{b}} \tag{93}$$

for any vector $\boldsymbol{b} \in \mathbb{R}^n$.

So all orthogonal projections are given by left multiplication with a suitable matrix (unoriginally called a "projection matrix"). The task for us now is to compute the matrix corresponding to a given projection, and we will do it in the special case when:

$$V = C(A)$$

for an $n \times m$ matrix A. In fact, this does not represent any restriction on the subspace V, since any subspace can be written as the column space of a suitably chosen matrix: just take basis vectors of the subspace and put them next to each other in a matrix. So given $\boldsymbol{b} \in \mathbb{R}^n$, we are looking for:

$$\boldsymbol{p} \in C(A) \quad \text{such that} \quad (\boldsymbol{b} - \boldsymbol{p}) \perp C(A)$$

But $\boldsymbol{p} \in C(A)$ precisely means:

$$\boldsymbol{p} = A\boldsymbol{v} \tag{94}$$

for some $m \times 1$ vector $\boldsymbol{v}$. Moreover, the condition $(\boldsymbol{b} - \boldsymbol{p}) \perp C(A)$ is equivalent to:

$$A^T(\boldsymbol{b} - \boldsymbol{p}) = 0 \tag{95}$$

as a product of matrices. Indeed, the i–th row of (95) is precisely the dot product $\boldsymbol{a}_i \cdot (\boldsymbol{b} - \boldsymbol{p})$, where $\boldsymbol{a}_i$ is the i–th column of A, and its vanishing means precisely that $\boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}_i$. Combining (94) with (95) gives us:

$$A^T(\boldsymbol{b} - A\boldsymbol{v}) = 0 \quad \Rightarrow \quad A^T A\boldsymbol{v} = A^T \boldsymbol{b}$$

Therefore, we may solve for $\boldsymbol{v} = (A^TA)^{-1}A^T\boldsymbol{b}$, and by combining this with (94) we obtain the following formula for the projection of the vector $\boldsymbol{b}$ onto the subspace $C(A)$:

$$\boxed{\text{proj}_{C(A)}\boldsymbol{b} = A(A^TA)^{-1}A^T\boldsymbol{b}} \tag{96}$$

Therefore, we conclude that the $n \times n$ projection matrix corresponding to the subspace $C(A)$ is:

$$\boxed{P_{C(A)} = A(A^TA)^{-1}A^T} \tag{97}$$

If the case $m = 1$, i.e. A is a single $n \times 1$ vector, formula (97) is equivalent to (92). The $m \times m$ matrix $S = A^TA$ which appears in (97) is symmetric, but we need it to be invertible in order for the formulas presented above to hold. We have the following fact:

$$\textbf{if the columns of } A \textbf{ are independent, then } S = A^TA \textbf{ is invertible} \tag{98}$$

Note that the equality:

$$(A^TA)^{-1} \neq A^{-1}(A^T)^{-1}$$

does not hold in general, because when A is a rectangular matrix, its inverse is not well-defined.

Let's do an example. Suppose we are in $n = 3$ dimensional space, and we want to compute the projection onto the plane V spanned by the vectors:

$$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{99}$$

First of all, let's make sure that these vectors are linearly independent (if one were a linear combination of the other, we could just throw it out). Then, we put the vectors together in a matrix:

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

As we saw in (97), an important role is played by the symmetric matrix:

$$S = A^TA = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

whose inverse (this matrix is invertible because the vectors (99) are independent) is given by:

$$S^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

Then formula (97) for the projection operator onto the subspace spanned by the vectors (99) is:

$$P_V = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} \tag{100}$$

We conclude that the projection of any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ onto the gives 2-dimensional plane is:

$$P_V \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5x - y - 2z \\ -x + 5y - 2z \\ -2x - 2y + 2z \end{bmatrix}$$

Remark. *Projection matrices are special in that they satisfy the equation:*

$$P_V P_V = P_V$$

for any subspace V. This corresponds to the geometric fact that if you projection of a projection is the projection itself. Check that this equation holds for the specific example (100).

Lecture 13 (*October 7*) *Reading: section 4.3*

One major computational application of projections is **least squares** approximation, which we will now delve into. Mathematically, the problem involves a system of equations for given A and $\boldsymbol{b}$:

$$A\boldsymbol{v} = \boldsymbol{b} \tag{101}$$

which does not have any solutions, because the number of equations is greater than the number of variables (mathematically, think of A as an $m \times n$ matrix with $m > n$). Therefore, the task is to find a value of $\boldsymbol{v}$ which makes $A\boldsymbol{v}$ as close as possible to $\boldsymbol{b}$. In other words, the **error** vector:

$$\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{v}$$

should be made as "small" as possible. Since we are dealing with vectors, "small" means that the length of the vector should be as small as possible. Therefore, the problem is to find $\boldsymbol{v}$ such that:

$$||\boldsymbol{b} - A\boldsymbol{v}|| \tag{102}$$

is as small as possible, for given A and $\boldsymbol{b}$. But what does this mean geometrically? As $\boldsymbol{v}$ ranges over all possible vectors, $A\boldsymbol{v}$ ranges over the column space $C(A)$. If $\boldsymbol{b}$ were already in $C(A)$, then the equation (101) could be satisfied with no error. But in general, $\boldsymbol{b} \notin C(A)$, in which case:

$$||\boldsymbol{b} - A\boldsymbol{v}|| \geq \Big(\text{distance from } \boldsymbol{b} \text{ to the subspace } C(A)\Big) = ||\boldsymbol{b} - \boldsymbol{p}||$$

where $\boldsymbol{p}$ is the orthogonal projection of $\boldsymbol{b}$ onto the subspace $C(A)$. Therefore, finding $\boldsymbol{v}$ which minimizes the quantity (102) actually involves two steps:

- find the orthogonal projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the subspace $C(A)$
- solve the equation $\boxed{A\boldsymbol{v} = \boldsymbol{p}}$ (which can be done, since $\boldsymbol{p} \in C(A)$)

We already know the answer to the first question from (97):

$$\boldsymbol{p} = A(A^TA)^{-1}A^T\boldsymbol{b}$$

The answer to the second question is also manifest, because a solution to the boxed equation is:

$$\boldsymbol{v} = (A^TA)^{-1}A^T\boldsymbol{b} \tag{103}$$

Recall the fact (98): in order for the matrix A^TA to be invertible and for the formula above to make sense, we need the matrix A to have independent columns. This can always be arranged by removing some dependent columns from the matrix A: for example, if the last column of A is a linear combination of the other ones, then we can simply remove the last column of A and the last entry of $\boldsymbol{v}$, and the task of approximating equation (101) as well as possible remains the same.

Remark. *You could also have arrived at formula* (103) *from calculus. Specifically, it's obvious that* $||\boldsymbol{b} - A\boldsymbol{v}||$ *is minimum when* $||\boldsymbol{b} - A\boldsymbol{v}||^2$ *is minimum. The usual formulas give us:*

$$||\boldsymbol{b} - A\boldsymbol{v}||^2 = (\boldsymbol{b} - A\boldsymbol{v})^T(\boldsymbol{b} - A\boldsymbol{v}) = \boldsymbol{b}^T\boldsymbol{b} - 2(A\boldsymbol{v})^T\boldsymbol{b} + (A\boldsymbol{v})^T(A\boldsymbol{v}) = \boldsymbol{b}^T\boldsymbol{b} - 2\boldsymbol{v}^TA^T\boldsymbol{b} + \boldsymbol{v}^TA^TA\boldsymbol{v}$$

Now, the latter quantity is minimum when its derivative with respect to $\boldsymbol{v}$ *vanishes (there is a mathematical notion of taking derivative with respect to a vector of variables, and it behaves just like derivative with respect to a single variable), which explicitly states that:*

$$0 = \frac{\partial(\boldsymbol{b}^T\boldsymbol{b} - 2\boldsymbol{v}^TA^T\boldsymbol{b} + \boldsymbol{v}^TA^TA\boldsymbol{v})}{\partial \boldsymbol{v}} = -2A^T\boldsymbol{b} + 2A^TA\boldsymbol{v}$$

Solving the equation above for $\boldsymbol{v}$ *gives precisely* (103).

Let's do a concrete example. Find numbers x, y, z such that the vector:

$$x\begin{bmatrix}1\\1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\1\end{bmatrix} + z\begin{bmatrix}1\\2\\1\end{bmatrix} \quad \text{is as close to} \quad \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{as possible} \tag{104}$$

A straightforward application of the analysis above would have us set $A = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 2\\0 & 1 & 1\end{bmatrix}$ and apply formula (103). But the columns of A are not independent: in fact, the third column is the sum of the first two. This is reflected in the fact that equation (104) reads:

$$x\begin{bmatrix}1\\1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\1\end{bmatrix} + z\left(\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}\right) = (x+z)\begin{bmatrix}1\\1\\0\end{bmatrix} + (y+z)\begin{bmatrix}0\\1\\1\end{bmatrix}$$

Since x, y, z can be chosen arbitrarily, the problem us equivalent to finding a and b such that:

$$\text{the vector} \quad a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} \quad \text{is as close to} \quad \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{as possible}$$

In matrix form, this is translated as:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ is as small as possible} \tag{105}$$

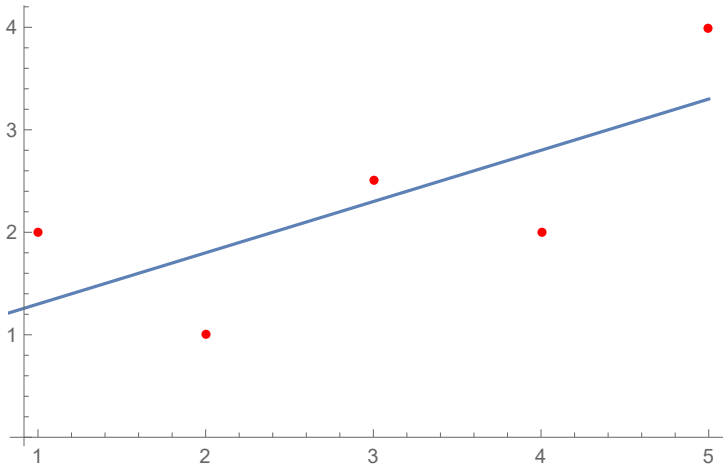
Now the columns are independent, so we can apply formula (103). We have:

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Rightarrow \quad (A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and so (103) implies that the difference (105) is minimized for:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{7}{3} \end{bmatrix}$$

Here is perhaps the standard application of least squares approximation, which will explain its name. Suppose you have a graph which seeks to represent a time-dependent quantity $y(t)$. You measure this quantity at times $t_1, ..., t_m$ and get values $y_1, ..., y_m$. Mark the data points (t_1, y_1),...,(t_m, y_m) on a graph, as follows (say $m = 5$ data points in the picture below):



The goal is to "fit" a straight line L between these data points as closely as possible. Mathematically, the word "closely" means that the following condition must be satisfied:

the sum of squares of the vertical distances from the data points to L is minimum

which explains the term "least squares". So how do we do find this best fit line? To choose a line is the same thing as to choose two numbers a and b, so that the equation of the line is $y(t) = a + bt$. Then the sum of squares that we need to minimize is the quantity:

$$(y_1 - a - bt_1)^2 + ... + (y_m - a - bt_m)^2 \tag{106}$$

where $t_1, ..., t_m$ and $y_1, ..., y_m$ are given to you by your experiment. How to convert this into a linear algebra problem? If the line were to pass precisely through our data points, this would require the

following system of equations to be satisfied (where a and b are the unknowns):

$$\begin{cases} a + bt_1 = y_1 \\ \dots \\ a + bt_m = y_m \end{cases} \quad \text{or, equivalently} \quad \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

In practice, we will not be able to solve this system on the nose, but the condition that the quantity (106) is minimized is exactly the condition that the square of (102) is minimized, for:

$$A = \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

We have:

$$A^T A = \begin{bmatrix} 1 & \dots & 1 \\ t_1 & \dots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ \dots & \dots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} m & t_1 + \dots + t_m \\ t_1 + \dots + t_m & t_1^2 + \dots + t_m^2 \end{bmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad (A^T A)^{-1} = \frac{1}{m(t_1^2 + \dots + t_m^2) - (t_1 + \dots + t_m)^2} \begin{bmatrix} t_1^2 + \dots + t_m^2 & -t_1 - \dots - t_m \\ -t_1 - \dots - t_m & m \end{bmatrix}$$

So formula (103) tells us that the choice of a and b which gives the best straight line fit is:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{m(t_1^2 + \dots + t_m^2) - (t_1 + \dots + t_m)^2} \begin{bmatrix} t_1^2 + \dots + t_m^2 & -t_1 - \dots - t_m \\ -t_1 - \dots - t_m & m \end{bmatrix} \begin{bmatrix} 1 & \dots & 1 \\ t_1 & \dots & t_m \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

and therefore:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{m(t_1^2 + \dots + t_m^2) - (t_1 + \dots + t_m)^2} \begin{bmatrix} \sum_{i=1}^n t_i(t_i - t_1) & \dots & \sum_{i=1}^n t_i(t_i - t_m) \\ \sum_{i=1}^n (t_1 - t_i) & \dots & \sum_{i=1}^n (t_m - t_i) \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

The fact that the fraction in the formula above is non-zero is a consequence of the Cauchy-Schwarz inequality, which states that $m(t_1^2 + \dots + t_m^2) > (t_1 + \dots + t_m)^2$ if $t_1, \dots, t_m$ are all distinct numbers.

Lecture 14 (*October 9*) *Reading: section 4.4*

Question: when are two planes V and W orthogonal? Answer: when any two vectors $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$ are orthogonal. Of course, it is hard to check this property for any vectors, but it suffices to check it for all vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ and $\boldsymbol{w}_1, \dots, \boldsymbol{w}_l$ which form bases of V and W, respectively:

$$V \perp W \quad \Leftrightarrow \quad \boldsymbol{v}_i^T \cdot \boldsymbol{w}_j = 0 \text{ for all } i \in \{1, \dots, k\},\ j \in \{1, \dots, l\} \tag{107}$$

Since there are k basis vectors of V and l basis vectors of W, there are $k \cdot l$ conditions to check above. You can replace this with a single condition, if you put the v's together in a matrix A and the w's together in a matrix B:

$$A = \left[\ \boldsymbol{v}_1 \ \middle|\ \dots \ \middle|\ \boldsymbol{v}_k\ \right] \qquad B = \left[\ \boldsymbol{w}_1 \ \middle|\ \dots \ \middle|\ \boldsymbol{w}_l\ \right]$$

Then $V = C(A)$ and $W = C(B)$, and condition (107) reads:

$$V \perp W \quad \Leftrightarrow \quad V^T W = 0 \tag{108}$$

The following is a related notion, which applies to bases of a single vector space V.

Definition 14. *A collection of non-zero vectors* $\boldsymbol{q}_1$,..., $\boldsymbol{q}_n$ *are called* ***orthogonal*** *if:*

$$\boldsymbol{q}_i \perp \boldsymbol{q}_j \quad \textit{or, equivalenty} \quad \boldsymbol{q}_i^T \boldsymbol{q}_j = 0 \quad \textit{for all } 1 \leq i \neq j \leq n \tag{109}$$

It is called ***orthonormal*** *if the vectors are both orthogonal and normalized to:*

$$||\boldsymbol{q}_i|| = 1 \qquad \textit{or, equivalenty} \qquad \boldsymbol{q}_i^T \boldsymbol{q}_i = 1 \quad \textit{for all } 1 \leq i \leq n \tag{110}$$

Orthogonal vectors are always linearly independent (so a basis of the vector space they span).

The standard basis $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ of $\mathbb{R}^n$ (where $\boldsymbol{e}_i = (0, ..., 0, 1, 0, ..., 0)$ with the 1 on the i–th spot) is orthonormal. In fact, the notion of orthonormality seeks to generalize this behavior to other bases.

There are n^2 conditions (109)–(110), so there are n^2 properties to check in order to show that a collection of n vectors are orthogonal. As before, this can be packaged into a single matrix equality, if we consider the $m \times n$ matrix (assume $\boldsymbol{q}_1, ..., \boldsymbol{q}_n \in \mathbb{R}^m$):

$$Q = \left[\begin{array}{c|c|c} & & \\ \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \\ & & \end{array} \right]$$

Then the orthogonality property (109) precisely states that:

$$Q^T Q = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} \tag{111}$$

where $d_i = ||\boldsymbol{q}_i||^2$. Therefore, the orthonormality property (109)–(110) precisely states that:

$$\boxed{Q^T Q = I_n} \tag{112}$$

In the particular case when $m = n$, we have:

Definition 15. *A square matrix* Q *is called* ***orthogonal*** *if* (112) *holds, i.e. if* $Q^T = Q^{-1}$.

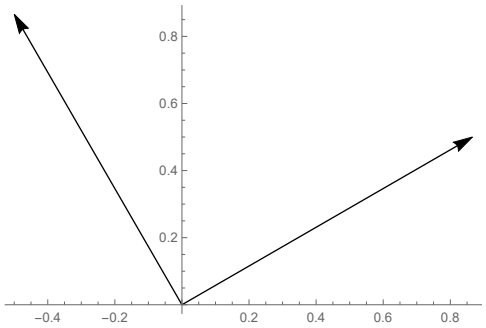
Examples of orthogonal matrices include all permutation matrices:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and the rotation matrices:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{113}$$

The columns of the matrix Q are the usual unit vectors in $\mathbb{R}^2$, rotated by an angle θ:



and they are orthonormal because $(\sin\theta)^2 + (\cos\theta)^2 = 1$. Both examples above share a feature, which is actually common of all orthogonal matrices: they preserve dot products, lengths and perpendicularity of vectors:

$$\text{if } Q \text{ is orthogonal, then } \boxed{(Q\boldsymbol{v})^T(Q\boldsymbol{w}) = \boldsymbol{v}^T\boldsymbol{w}} \tag{114}$$

for all vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. The reason for this is simply that $(Q\boldsymbol{v})^T(Q\boldsymbol{w}) = \boldsymbol{v}^T Q^T Q\boldsymbol{w} = \boldsymbol{v}^T I\boldsymbol{w} = \boldsymbol{v}^T\boldsymbol{w}$. Therefore, as a consequence, we have:

$$\text{if } \boldsymbol{v} \perp \boldsymbol{w} \quad \text{then} \quad Q\boldsymbol{v} \perp Q\boldsymbol{w} \tag{115}$$

and:

$$||Q\boldsymbol{v}|| = ||\boldsymbol{v}|| \tag{116}$$

Now let's return to the projection formula (97), which is generally cumbersome because we need to invert the matrix A^TA. In the particular case when $A = Q$ has orthonormal columns (i.e. we have an orthonormal basis of the vector space we are projecting upon), the projection formula is nicer:

$$P_{C(Q)} = QQ^T \tag{117}$$

because $Q^TQ = I$. Similarly, in this case the solution to the least squares problem (103) simplifies to $\boldsymbol{v} = Q^T\boldsymbol{b}$. In words: if you're projecting onto the subspace with general basis vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$, then you need to apply the more complicated formula (97) for $A = \big[\boldsymbol{v}_1|...|\boldsymbol{v}_n\big]$. But if you know that your subspace has a basis consisting of orthonormal vectors $\boldsymbol{q}_1, ..., \boldsymbol{q}_n$, then you may apply the simpler formula (117) for $Q = \big[\boldsymbol{q}_1|...|\boldsymbol{q}_n\big]$. So it pays to have orthonormal bases of subspaces.

Fortunately, not only do all subspaces have orthonormal bases, there's an explicit algorithm which allows you to construct one of these from any given basis. This is called the **Gram-Schmidt process** and here's how it goes: you start from a general collection of linearly independent vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ and the goal is to "make" them orthonormal. You are allowed to modify the individual vectors $\boldsymbol{v}_i$ by rescaling them and adding linear combinations of the other vectors:

$$\boldsymbol{v}_i \quad \rightsquigarrow \quad \alpha_1\boldsymbol{v}_1 + ... + \alpha_i\boldsymbol{v}_i + ... + \alpha_n\boldsymbol{v}_n \quad \text{with } \alpha_i \neq 0$$

since these operations do not change the subspace spanned by $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$. Start by renormalizing $\boldsymbol{v}_1$ so that it has length 1:

$$\text{first step:} \quad \boldsymbol{v}_1 \rightsquigarrow \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{||\boldsymbol{v}_1||}$$

Then modify $\boldsymbol{v}_2$ so that it is perpendicular to $\boldsymbol{q}_1$, and then renormalize it to have length 1:

$$\text{second step:} \quad \boldsymbol{v}_2 \rightsquigarrow \boldsymbol{w}_2 = \boldsymbol{v}_2 - \text{proj}_{\boldsymbol{q}_1} \boldsymbol{v}_2 \quad \text{and} \quad \boldsymbol{w}_2 \rightsquigarrow \boldsymbol{q}_2 = \frac{\boldsymbol{w}_2}{||\boldsymbol{w}_2||}$$

Then modify $\boldsymbol{v}_3$ so that it is perpendicular to $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, and then renormalize it to have length 1:

$$\text{third step:} \quad \boldsymbol{v}_3 \rightsquigarrow \boldsymbol{w}_3 = \boldsymbol{v}_3 - \text{proj}_{\boldsymbol{q}_1} \boldsymbol{v}_3 - \text{proj}_{\boldsymbol{q}_2} \boldsymbol{v}_3 \quad \text{and} \quad \boldsymbol{w}_3 \rightsquigarrow \boldsymbol{q}_3 = \frac{\boldsymbol{w}_3}{||\boldsymbol{w}_3||}$$

... so far and so forth, until the last step where we modify $\boldsymbol{v}_n$ so that it is perpendicular to $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, ..., $\boldsymbol{q}_{n-1}$ and then renormalize it to have length 1:

$$\text{last step:} \quad \boldsymbol{v}_n \rightsquigarrow \boldsymbol{w}_n = \boldsymbol{v}_n - \text{proj}_{\boldsymbol{q}_1} \boldsymbol{v}_n - ... - \text{proj}_{\boldsymbol{q}_{n-1}} \boldsymbol{v}_n \quad \text{and} \quad \boldsymbol{w}_n \rightsquigarrow \boldsymbol{q}_n = \frac{\boldsymbol{w}_n}{||\boldsymbol{w}_n||}$$

At the end of this procedure, all the vectors $\boldsymbol{q}_1, ..., \boldsymbol{q}_n$ are orthogonal to each other by construction, and they all have length 1. This is because at each step, we subtract from every $\boldsymbol{v}_i$ its projection onto the already constructed vectors $\boldsymbol{q}_1, ..., \boldsymbol{q}_{i-1}$. What remains will be orthogonal to these vectors. The various projections that one must calculate in this process, namely $\text{proj}_{\boldsymbol{q}_j} \boldsymbol{v}_i$ can be computed using formula (89) (which is even simpler in our case, since the denominator of (89) is equal to 1).

Let's see what Gram-Schmidt tells us about the relation between the matrix:

$$A = \left[\begin{array}{c|c|c} \boldsymbol{v}_1 & \dots & \boldsymbol{v}_n \end{array} \right] \quad \text{and} \quad Q = \left[\begin{array}{c|c|c} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{array} \right]$$

where $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are the input vectors and $\boldsymbol{q}_1, ..., \boldsymbol{q}_n$ are the output vectors of Gram-Schmidt. At each step, we are modifying the i–th column of a matrix by subtracting from it a linear combination of the previous columns, and then multiplying it by a scalar. This is doing the same thing for columns as row operations were doing for rows. Therefore, it shouldn't surprise you that:

- Adding the j–th column times λ to the i–th column of A is achieved by multiplying the latter on the right with an elimination matrix:

$$A \rightsquigarrow AE_{ji}^{(\lambda)} \qquad \text{where} \qquad E_{ji}^{(\lambda)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  The entry λ is on the j–th row and i–th column (in the formula above, $i = 4$ and $j = 2$).

- Multiplying the i–th column of A by the scalar λ is achieved by multiplying the latter on the right with a diagonal matrix:

$$A \rightsquigarrow AD_{i}^{(\lambda)} \qquad \text{where} \qquad D_{i}^{(\lambda)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  The entry λ is on i–th row (in the formula above, $i = 2$).

Compare these with the operations you learned in Lecture 2 when discussing Gaussian elimination and LU factorizations: the upshot is that while row operations correspond to multiplication on the left, column operations correspond to multiplication on the right. Therefore, we conclude that the connection between the input and the output of Gram-Schmidt is:

$$Q = A \underbrace{D_1^{(\mu_1)}}_{\text{first step}} \underbrace{E_{12}^{(\lambda_{12})} D_2^{(\mu_2)}}_{\text{second step}} \underbrace{E_{13}^{(\lambda_{13})} E_{23}^{(\lambda_{23})} D_3^{(\mu_3)}}_{\text{third step}} ... \underbrace{E_{1n}^{(\lambda_{1n})} ... E_{n-1,n}^{(\lambda_{n-1,n})} D_n^{(\mu_n)}}_{\text{last step}} \tag{118}$$

where the scalars μ_i and λ_{ij} are determined by the orthonomality requirement at every stage of Gram-Schmidt. By moving all the E's and D's in the other side of the equation, we obtain:

$$A = Q D_n^{\left(\frac{1}{\mu_n}\right)} E_{n-1,n}^{(-\lambda_{n-1,n})} ... E_{1n}^{(-\lambda_{1n})} ... D_3^{\left(\frac{1}{\mu_3}\right)} E_{23}^{(-\lambda_{23})} E_{13}^{(-\lambda_{13})} D_2^{\left(\frac{1}{\mu_2}\right)} E_{12}^{(-\lambda_{12})} D_1^{\left(\frac{1}{\mu_1}\right)}$$

If we multiply the D and E matrices together, we will obtain an upper triangular matrix R. Hence:

Fact 10. *Any matrix A with linearly independent columns can be uniquely written as:*

$$\boxed{A = QR} \tag{119}$$

where the columns of Q are orthonormal and R is an upper triangular square matrix:

$$R = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

Lecture 15 (*October 11*) *Reading: section 8.1*

We will now introduce the ultimate level of abstraction in our course, which will finally explain the meaning of matrix multiplication. Recall n–dimensional space, which as a set is given by:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_1, ..., x_n \in \mathbb{R} \right\}$$

Definition 16. *A **linear transformation** is a function:*

$$\phi : \mathbb{R}^n \to \mathbb{R}^m \tag{120}$$

which respects the vector space structures of $\mathbb{R}^n$ and $\mathbb{R}^m$, namely addition and scalar multiplication:

$$\phi(\boldsymbol{v} + \boldsymbol{v}') = \phi(\boldsymbol{v}) + \phi(\boldsymbol{v}') \qquad \text{and} \qquad \phi(\alpha \cdot \boldsymbol{v}) = \alpha \cdot \phi(\boldsymbol{v}) \tag{121}$$

for any vectors $\boldsymbol{v}, \boldsymbol{v}' \in \mathbb{R}^n$ and any scalar $\alpha \in \mathbb{R}$.

Let us consider a linear transformation ϕ as in (120), and express it in terms of the basis:

$$\boldsymbol{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

(whose only entry 1 is on the j–th row). The value $\phi(\boldsymbol{e}_j)$ will be a vector in $\mathbb{R}^m$, and therefore a linear combination of the basis vectors $\boldsymbol{e}_i$. This means that there exist constants $a_{ij} \in \mathbb{R}$ such that:

$$\phi(\boldsymbol{e}_j) = \sum_{i=1}^{m} a_{ij}\boldsymbol{e}_i \qquad \forall\ j \in \{1, ..., n\} \tag{122}$$

But once you know the numbers a_{ij}, the linear transformation ϕ is uniquely determined! Indeed, since any vector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is a linear combination of the $\boldsymbol{e}_i$'s, formulas (121) and (122) imply:

$$\phi(\boldsymbol{v}) = \phi(v_1\boldsymbol{e}_1 + ... + v_n\boldsymbol{e}_n) = \sum_{j=1}^{n} v_j\phi(\boldsymbol{e}_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}v_j\boldsymbol{e}_i$$

which completely determines $\phi(\boldsymbol{v})$ for any $\boldsymbol{v}$. In column matrix form, this relation reads:

$$\phi\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + ... + a_{1n}v_n \\ \vdots \\ a_{m1}v_1 + ... + a_{mn}v_n \end{bmatrix} \tag{123}$$

Does this look familiar? Indeed, ϕ acts on column vectors precisely like the matrix:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \tag{124}$$

multiplies vectors. We conclude the following:

Fact 11. *There is a one-to-one correspondence between linear transformations and matrices, in that the matrix* (124) *represents the linear transformation* (123)*. In symbols, this means:*

$$\boxed{\phi(\boldsymbol{v}) = A\boldsymbol{v}} \tag{125}$$

for all vectors $\boldsymbol{v}$*.*

Let us discuss the 2×2 case in detail, by showing which geometrically relevant linear transformations $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ correspond to which matrices:

- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is reflection in the line $\{x = y\}$
- $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is projection onto the x-axis
- $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ is a scaling by a factor of λ in the x direction and by a factor of μ in the y direction

- $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is counter-clockwise rotation by an angle of θ around the origin
- $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ is a shearing parallel to the x axis

As you can see from these examples, linear transformations send lines to lines and geometric shapes of a certain kind (triangles, quadrilaterals, ellipses) to shapes of the same kind. Another big class of linear transformations are projections, and formula (96) basically states that the function:

$$\text{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$$

which projects points onto a given subspace V corresponds to the matrix:

$$A(A^TA)^{-1}A^T$$

where the columns of the $n \times m$ matrix A are given by any basis of V.

Note that composition of linear transformations corresponds to multiplication of matrices. In symbols, if the linear transformations ϕ_1, ϕ_2 are given by matrices A_1, A_2, respectively, then:

$$\boxed{\text{the transformation } \phi_1 \circ \phi_2 \text{ corresponds to the matrix } A_1A_2} \tag{126}$$

For example, rotation by $\frac{\pi}{2}$ degrees followed by a horizontal shearing is given by the matrix:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Take now $m = n$. If we have a function $\phi : \mathbb{R}^n \to \mathbb{R}^n$, we can talk about its **inverse** $\phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$: the latter is connected to the former by the condition that if $\phi(\boldsymbol{v}) = \boldsymbol{w}$, then $\phi^{-1}(\boldsymbol{w}) = \boldsymbol{v}$. It may not surprise you that if the linear transformation ϕ corresponds to a square matrix A, then:

$$\boxed{\text{the transformation } \phi^{-1} \text{ corresponds to the matrix } A^{-1}} \tag{127}$$

The function ϕ has an inverse precisely if and only if the matrix A is invertible.

The nullspace and column space of a matrix A also have an interpretation in terms of the corresponding linear transformation $\phi : \mathbb{R}^n \to \mathbb{R}^m$:

$$N(A) = \text{Ker } \phi \qquad \text{and} \qquad C(A) = \text{Im } \phi \tag{128}$$

where the kernel Ker ϕ is the set of vectors $\boldsymbol{v} \in \mathbb{R}^n$ such that $\phi(\boldsymbol{v}) = 0$, and the image Im ϕ is the set of vectors $\boldsymbol{w} \in \mathbb{R}^m$ which can be written as $\phi(\boldsymbol{v})$ for some vector $\boldsymbol{v} \in \mathbb{R}^n$.

Lecture 16 (*October 16*) *Reading: section 8.2*

In our last lecture, we said that a linear transformation $\phi : \mathbb{R}^n \to \mathbb{R}^m$ corresponds to an $m \times n$ matrix A. However, this is not completely precise: while a linear transformation ϕ is an intrinsic

geometric notion (namely a rule that sends points to points and lines to lines) the matrix A is simply a representation of it. For example, let the linear transformation:

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{be counterclockwise rotation by } 30^\circ$$

The matrix $A = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ is a representation of this rotation in the usual (x, y) system of coordinates, i.e. in the usual basis $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of $\mathbb{R}^2$. Indeed, all that this is saying is that:

$$\phi(x \cdot \boldsymbol{e}_1 + y \cdot \boldsymbol{e}_2) = x \cdot \boldsymbol{a}_1 + y \cdot \boldsymbol{a}_2 = \left(\frac{x\sqrt{3}}{2} - \frac{y}{2}\right) \cdot \boldsymbol{e}_1 + \left(\frac{x}{2} + \frac{y\sqrt{3}}{2}\right) \cdot \boldsymbol{e}_2 \tag{129}$$

where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are the columns of A. But now suppose you wanted to express this transformation in a different reference frame, i.e. changing the basis from $\boldsymbol{e}_1, \boldsymbol{e}_2$ to $\boldsymbol{v}_1 = 2\boldsymbol{e}_1$, $\boldsymbol{v}_2 = \boldsymbol{e}_1 - \boldsymbol{e}_2$. When you change basis from $\boldsymbol{e}$'s to $\boldsymbol{v}$'s, you can always change back, i.e. express the $\boldsymbol{e}$'s in terms of $\boldsymbol{v}$'s: in our case we have $\boldsymbol{e}_1 = \frac{\boldsymbol{v}_1}{2}$, $\boldsymbol{e}_2 = \frac{\boldsymbol{v}_1}{2} - \boldsymbol{v}_2$. We may plug these formulas into (129) and we get:

$$\phi(x \cdot \boldsymbol{v}_1 + y \cdot \boldsymbol{v}_2) = \phi(x \cdot 2\boldsymbol{e}_1 + y \cdot (\boldsymbol{e}_1 - \boldsymbol{e}_2)) = \phi((2x + y) \cdot \boldsymbol{e}_1 - y \cdot \boldsymbol{e}_2) \overset{(129)}{=} \tag{130}$$

$$\left[\frac{(2x+y)\sqrt{3}}{2} + \frac{y}{2}\right]\boldsymbol{e}_1 + \left[\frac{2x+y}{2} - \frac{y\sqrt{3}}{2}\right]\boldsymbol{e}_2 = \left[\frac{(2x+y)\sqrt{3}}{2} + \frac{y}{2}\right]\frac{\boldsymbol{v}_1}{2} + \left[\frac{2x+y}{2} - \frac{y\sqrt{3}}{2}\right]\left(\frac{\boldsymbol{v}_1}{2} - \boldsymbol{v}_2\right)$$

$$= \left[x \cdot \frac{\sqrt{3}+1}{2} + y \cdot \frac{1}{2}\right] \cdot \boldsymbol{v}_1 + \left[x \cdot (-1) + y \cdot \frac{\sqrt{3}-1}{2}\right] \cdot \boldsymbol{v}_2$$

So in terms of the new basis $\boldsymbol{v}_1, \boldsymbol{v}_2$, the linear transformation ϕ is represented by the matrix:

$$B = \begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{1}{2} \\ -1 & \frac{\sqrt{3}-1}{2} \end{bmatrix}$$

The connection between this matrix and the original matrix A is precisely:

$$\boxed{B = V^{-1}AV} \tag{131}$$

where $V = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ is the change of basis matrix from the basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ to the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$ (specifically, it is the matrix whose columns are the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$).

So we have seen that the matrices A and B connected by (131) describe the same linear transformation, albeit in different bases. Let us explain the general principle for any linear transformation:

$$\phi : \mathbb{R}^n \to \mathbb{R}^n$$

which is represented by the $n \times n$ matrix $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ in the standard basis:

$$\phi(x_1\boldsymbol{e}_1 + ... + x_n\boldsymbol{e}_n) = (a_{11}x_1 + ... + a_{1n}x_n)\boldsymbol{e}_1 + ... + (a_{n1}x_1 + ... + a_{nn}x_n)\boldsymbol{e}_n \tag{132}$$

If you wish to express the linear transformation ϕ in a different basis $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$, then you need to consider the **conjugate** matrix:

$$B = V^{-1}AV = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \quad \text{where} \quad V = \left[\begin{array}{c|c|c} & & \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_n \\ & & \end{array} \right]$$

In terms of the new basis, formula (132) may be rewritten as:

$$\phi(x_1\boldsymbol{v}_1 + ... + x_n\boldsymbol{v}_n) = (b_{11}x_1 + ... + b_{1n}x_n)\boldsymbol{v}_1 + ... + (b_{n1}x_1 + ... + b_{nn}x_n)\boldsymbol{v}_n \tag{133}$$

Example 3. *The projection onto the vector* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *in* $\mathbb{R}^2$ *is represented by the matrix:*

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

in the usual $\boldsymbol{e}_1, \boldsymbol{e}_2$ *basis of the plane. But suppose you wanted to represent the same linear transformation in the basis* $\boldsymbol{v}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2$, $\boldsymbol{v}_2 = \boldsymbol{e}_1 - \boldsymbol{e}_2$ *instead. The way you would do this is encode the change of basis into the matrix:*

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then formula (131) *says that the given projection linear transformation is represented by the matrix:*

$$B = V^{-1}AV = \left(-\frac{1}{2} \right) \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

It's not a coincidence that the answer is such a simple matrix. Indeed, all that the formula for B *is saying is that:*

$$\text{proj}_{(1,1)}(x \cdot \boldsymbol{v}_1 + y \cdot \boldsymbol{v}_2) = x \cdot \boldsymbol{v}_1$$

Geometrically, this says that projection sends $\boldsymbol{v}_1$ *to itself (i.e. leaves it unchanged, which makes sense given that* $\boldsymbol{v}_1$ *is precisely the vector being projected upon) and sends* $\boldsymbol{v}_2$ *to 0 (which makes sense given that* $\boldsymbol{v}_1 \perp \boldsymbol{v}_2$*).*

Finally, let us consider a general linear transformation:

$$\phi : \mathbb{R}^n \to \mathbb{R}^m$$

which is represented by the $m \times n$ matrix $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$ in the standard basis:

$$\phi(x_1\boldsymbol{e}_1 + ... + x_n\boldsymbol{e}_n) = (a_{11}x_1 + ... + a_{1n}x_n)\boldsymbol{e}_1 + ... + (a_{m1}x_1 + ... + a_{mn}x_n)\boldsymbol{e}_m$$

Since $\mathbb{R}^m$ and $\mathbb{R}^n$ are (in general) different vector spaces, it will often make sense to change the basis in the input $\mathbb{R}^n$ independently of the change of basis in the output $\mathbb{R}^m$:

$$\boldsymbol{e}_1, ..., \boldsymbol{e}_n \rightsquigarrow \boldsymbol{v}_1, ..., \boldsymbol{v}_n \text{ are bases of } \mathbb{R}^n$$
$$\boldsymbol{e}_1, ..., \boldsymbol{e}_m \rightsquigarrow \boldsymbol{w}_1, ..., \boldsymbol{w}_m \text{ are bases of } \mathbb{R}^m$$

If you wish to express the linear transformation ϕ in these new bases, consider the matrices:

$$V = \left[\begin{array}{c|c|c} \boldsymbol{v}_1 & \dots & \boldsymbol{v}_n \end{array} \right] \qquad \text{and} \qquad W = \left[\begin{array}{c|c|c} \boldsymbol{w}_1 & \dots & \boldsymbol{w}_m \end{array} \right]$$

Then the linear transformation ϕ is represented by the matrix:

$$B = W^{-1}AV = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} \tag{134}$$

in these new bases, by which we mean that:

$$\phi(x_1\boldsymbol{v}_1 + ... + x_n\boldsymbol{v}_n) = (b_{11}x_1 + ... + b_{1n}x_n)\boldsymbol{w}_1 + ... + (b_{m1}x_1 + ... + b_{mn}x_n)\boldsymbol{w}_m \tag{135}$$

The ability to change bases gives us a lot of freedom in studying matrices. Basically, you can always change bases in order to simplify any given matrix, and how far you go depends on the problem at hand. For example, in the square $n \times n$ case, <u>almost</u> any matrix is conjugate to a diagonal matrix:

$$A = V \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} V^{-1} \tag{136}$$

In future weeks, we will see the meaning of the numbers $d_1, ..., d_n$ (the eigenvalues of A) and of the columns of the matrix V (the eigenvectors of A). We will also learn how to eliminate the word "almost" in the previous paragraph, by studying Jordan normal forms.

Lecture 17 (*October 18)* *Reading: section 5.1*

Consider any linear transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$, which is represented by an $n \times n$ square matrix A.

Definition 17. *The **determinant** of the square matrix A is the number:*

$$\det A \tag{137}$$

defined as the factor by which the linear transformation ϕ scales the n–dimensional volumes of regions in $\mathbb{R}^n$ (the determinant can be negative if ϕ switches left-handedness to right-handedness).

Let's take for instance the linear transformations of $\mathbb{R}^2$ considered immediately after Fact 11. Two-dimensional volume just means area. Any reflection preserves the magnitude of areas but switches handedness, hence:

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \tag{138}$$

Projections flatten regions $R \subset \mathbb{R}^2$ to line segments, which have area 0, so therefore:

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{139}$$

Scaling transformations increase areas by the product of the scaling factors, and therefore:

$$\det \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} = \lambda\mu \tag{140}$$

Finally, rotations and shearings preserve areas, so we have:

$$\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 1$$

$$\det \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1 \tag{141}$$

In general, the determinant of a 2×2 matrix is given by the following formula:

$$\boxed{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc} \tag{142}$$

and indeed, the determinant plays an important role in the formula for the inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{143}$$

But before we work out general means for computing determinants, let us spell out two key facts.

Fact 12. *For any $n \times n$ matrices A and B, we have:*

$$\det AB = (\det A)(\det B) \tag{144}$$

Let's give a quick proof of this fact: if A and B correspond to linear transformations ϕ and ψ, respectively, then we have seen in (126) that AB corresponds to the composition $\phi \circ \psi$ of the two transformations. Then (144) is simply saying that the rate at which $\phi \circ \psi$ scales volumes is equal to the product of rates at which ϕ and ψ individually scale volumes. Indeed, if you take any region $R \subset \mathbb{R}^n$ and assume that:

$$\begin{cases} \psi \text{ takes } R \text{ to some region } R' \\ \phi \text{ takes } R' \text{ to some region } R'' \end{cases} \quad \Rightarrow \quad \phi \circ \psi \text{ takes } R \text{ to the region } R''$$

With this in mind, we have:

$$\text{LHS of (144)} = \frac{\text{vol } R''}{\text{vol } R} = \frac{\text{vol } R''}{\text{vol } R'} \cdot \frac{\text{vol } R'}{\text{vol } R} = \text{RHS of (144)}$$

which proves (144). Caveat: it is not true that $\det(A + B) = \det A + \det B$, in general!

Fact 13. *Row operations have the following effect on determinants:*

- *adding a multiple of one row to another leaves the determinant unchanged*

- *exchanging two rows multiplies the determinant by* -1
- *multiplying a single row by* λ *has the effect of multiplying the determinant by* λ

The analogous properties still hold if the word "row" is replaced by "column".

This fact actually follows from Fact 12, as we will now proceed to show. Recall from Lecture 2 that performing the three kinds of row operations above on a matrix A is precisely achieved by multiplying A on the left by elimination/permutation/diagonal matrices. Then the statements in the bullets of Fact 13 become equivalent to:

$$\begin{aligned}
\det E_{ij}^{(\lambda)} A = \det A \quad &\text{which happens because} \quad \det E_{ij}^{(\lambda)} = 1 \\
\det P_{ij} A = -\det A \quad &\text{which happens because} \quad \det P_{ij} = -1 \\
\det D_i^{(\lambda)} A = \lambda \cdot \det A \quad &\text{which happens because} \quad \det D_i^{(\lambda)} = \lambda
\end{aligned}$$

The fact that P_{ij}, $D_i^{(\lambda)}$, $E_{ij}^{(\lambda)}$ have determinants -1, λ, 1 respectively, is simply a natural generalization of (138), (140), (141). Fact 13 has the following important consequence:

$$\textbf{If } A \textbf{ is a square matrix, then } \boxed{\det A = \pm(\text{product of pivots of REF}(A))} \tag{145}$$

which means that a computationally efficient way to compute the determinant of a matrix is to just put it in row echelon form and multiply its pivots. Indeed, Fact 13 implies that $\det A = \pm \det U$, where U is the row echelon form of A, and the sign $\pm$ is simply (-1) raised to the number of row exchanges required by Gaussian elimination. So it remains to show that $\det U$ is the product of its pivots, which by applying the third bullet of Fact 13, is equivalent to showing that a upper triangular matrix with all 1's on the diagonal has determinant 1. This happens because such an upper diagonal matrix can be transformed into the unit matrix I by row operations (as in Gauss-Jordan elimination) and the unit matrix has determinant 1.

Another way to restate the discussion in the previous paragraph is to remember that any square matrix A has a factorization of the form:

$$PA = LDU$$

where L and U are lower/upper triangular with all 1's on the diagonal, and D is a diagonal matrix. Applying determinant and Fact 12 to the formula above gives us;

$$(\det P)(\det A) = (\det L)(\det D)(\det U)$$

and then (145) follows from the fact that $\det P = (-1)$ raised to the number of row exhanges, while $\det L = \det U = 1$ and $\det D =$ product of pivots. The latter formulas are instances of the following general formulas for the determinants of diagonal and triangular matrices:

$$\det \begin{bmatrix} d_1 & 0 & \dots & 0 \\ * & d_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & d_n \end{bmatrix} = \det \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} = \det \begin{bmatrix} d_1 & * & \dots & * \\ 0 & d_1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} = d_1 d_2 ... d_n \tag{146}$$

which are justified as in the paragraph after (145).

If a square matrix A has a full row of zeroes (or a full column of zeroes, for that matter), then its determinant is 0: this is simply a consequence of (145) and the fact that one of the pivots would be 0 if we had a full row of zeroes. But we also have the following more general property:

$$\textbf{If the rows/columns of } A \textbf{ are linearly dependent, then } \det A = 0 \tag{147}$$

Indeed, if the rows were dependent, then one of them (say row i) would be a linear combination of the other ones. But then by successively subtracting multiples of the other rows from row i, which we know do not change the determinant, we would arrive at a matrix with a full row of zeroes. Since such a matrix would have a zero pivot, it has determinant 0, thus establishing (147).

We know exactly when a matrix has linearly dependent rows/columns: precisely when it is singular, i.e. not invertible. Therefore, (147) can be restated as:

Fact 14. *A square matrix A is non-singular (invertible) if and only if* $\det A \neq 0$.

Indeed, if the matrix A had an inverse A^{-1}, then applying determinant and Fact 12 to the formula $AA^{-1} = I$ would imply that:

$$(\det A)(\det A^{-1}) = 1 \quad \Rightarrow \quad \det A^{-1} = \frac{1}{\det A} \tag{148}$$

This immediately implies that the determinant of an invertible matrix cannot be 0. Finally, we give one more fact for the road:

Fact 15. *For any square matrix A, we have* $\det A = \det A^T$.

If Q is an orthogonal matrix (see Definition 15), i.e. $Q^T = Q^{-1}$, then (148) and Fact 15 imply that its determinant can only be 1 or -1.

Lecture 18 (*October 21*) *Reading: section 5.2*

Formula (145) computes the determinant of a square matrix from its pivots, which is just what a computer would use. But there is a different, the so-called "big formula" for the determinant, which is very important for understanding the theory and properties of the determinant.

Let's start from the following observation: determinants are linear functions of each row separately. As a formula, this means that for any numbers x_{ij} and $a_1, ..., a_n$, $b_1, ..., b_n$, we have:

$$\det \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 + b_1 & a_2 + b_2 & \dots & a_n + b_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix} = \det \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix} + \det \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix} \tag{149}$$

Very importantly, the matrix on the left is **NOT** the sum of the matrices on the right. Instead, all three matrices have the same elements on all rows except for the i–th one (that's the one where the a's and b's are in the formula), and i could be anything. Moreover, we have:

$$\det \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_1 & \lambda a_2 & \dots & \lambda a_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix} = \lambda \det \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix} \tag{150}$$

Remark. *Properties* (149) *and* (150) *also hold if we replace rows by columns (so the matrices in the left and right-hand sides only differ on a single column) which makes sense given the fact that determinant is left unchanged by transposing matrices.*

Here's an argument for (149). You can perform row exchanges in order to make $i = n$, i.e. the three matrices only differ on the last row. Perform Gaussian elimination on the three matrices in (149). It's not hard to see that the first $n-1$ rows will look the same in all three matrices, so they will all have the same pivots $p_1, ..., p_{n-1}$. When you finally get to do Gaussian elimination on the n–th row, denote the pivots on the n–th rows of the matrices in the right-hand side of (149) by α and β, respectively. Then a little thought shows that the pivot on the n–th row of the matrix in the left-hand side of (149) has to be $\alpha + \beta$. Then formula (145) implies that (149) boils down to:

$$\pm p_1 ... p_{n-1}(\alpha + \beta) = \pm p_1 ... p_{n-1}\alpha \pm p_1 ... p_{n-1}\beta$$

which is definitely a true statement. Formula (150) is proved by a similar argument.

By applying formula (150) to all the rows of an $n \times n$ matrix, we get:

$$\det(\lambda A) = \lambda^n \cdot \det A \tag{151}$$

which you could also get by applying (144) to the matrix $B = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix}$.

Let's return to computing determinants. Since $\det(A + B) \neq \det A + \det B$ in general, we must resort to formula (149) in order to break up a general matrix into simpler, more manageable pieces. Let's take the 2×2 case for illustration. By applying (149) to the first row, we have:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then apply (149) to the second rows of each of the matrices in the right-hand side:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} + \det \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

The first and fourth determinants in the right-hand side vanish because of (147), since they have full 0 columns. Therefore, we are left with:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det \underline{\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}} + \det \underline{\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}} = a_{11}a_{22} - a_{12}a_{21}$$

Indeed, the first underlined matrix is already in row echelon form, so its determinant is the product of the two pivots. Meanwhile, the second underlined matrix would be in row echelon form as soon as we exchange the two rows, which is responsible for the fact that its determinant has a -1 sign in front. Thus, we have proved formula (142).

In general, the procedure of successively applying (149) to the rows of the matrix leaves us with a sum of many determinants, all of which have a single non-zero entry on each row. Of all those determinants, the ones which have a full zero column will vanish, which means that the only surviving determinants are the ones where the non-zero elements are on both different rows and different columns. For example, in the 3×3 case, we have:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix}$$
$$+ \det \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

There are as many surviving determinants as possible permutations of the rows, and indeed, all of these determinants can be transformed into diagonal matrices by an appropriate row exchange. This leads to the following formula:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \quad (152)$$

In general, the procedure gives a formula for the determinant as a sum over all permutations:

$$\{\sigma(1), ..., \sigma(n)\} \quad \text{of} \quad \{1, ..., n\}$$

and we have the following **big formula for the determinant**:

$$\det \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} = \sum_{\{\sigma(1),...,\sigma(n)\}}^{\text{permutations}} (-1)^{\text{sgn } \sigma} a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)} \quad (153)$$

The number sgn σ is called the **signature** of the permutation σ: it is the minimum number of successive (e.g. the i-th and $i+1$-th) row exchanges that you need to apply to the permutation matrix corresponding to σ, in order to turn it into the unit matrix.

Formula (153) is nice and explicit, but awful in practice. The number of summands for an $n \times n$ matrix is equal to the number of permutations of $\{1, ..., n\}$, which is $n! = 1 \cdot 2 \cdot ... \cdot n$. But there are matrices, particularly those which have a lot of zeroes, for which this formula is quite manageable. For example, if the i–th row only has a single non-zero entry (say on the j–th column) then we get:

$$\det \begin{bmatrix} \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ 0 & \dots & 0 & a_{ij} & 0 & \dots & 0 \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} =$$

$$= \sum_{\substack{\text{permutations} \\ \{\sigma(1),...,\sigma(i-1),\sigma(i+1),...,\sigma(n)\} \\ \text{of } \{1,...,j-1,j+1,...,n\}}} (-1)^{\text{signature}} a_{1\sigma(1)}...a_{i-1,\sigma(i-1)}a_{ij}a_{i+1,\sigma(i+1)}...a_{n\sigma(n)}$$

$$= a_{ij} \cdot (-1)^{i+j} \underbrace{\det \begin{bmatrix} \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}}_{\text{cofactor } C_{ij}} \tag{154}$$

This is because the only terms which give a non-zero contribution to (153) are those with $\sigma(i) = j$. So computing an $n \times n$ determinant whose only non-zero entry on row i is on column j boils down to computing the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j.

Formula (154) leads to a general way of computing determinants, known as **cofactor expansion**. Explicitly, for a matrix $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ and any row i, we have:

$$\det A \overset{(149)}{=} \sum_{j=1}^{n} \det \begin{bmatrix} \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ 0 & \dots & 0 & a_{ij} & 0 & \dots & 0 \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

We may invoke formula (154) to compute the right-hand side, and we obtain:

$$\boxed{\det A = a_{i1} \cdot C_{i1} + ... + a_{in} \cdot C_{in}} \tag{155}$$

where C_{ij} is called the (i,j)–**cofactor**:

$$C_{ij} = (-1)^{i+j} \det M_{ij} \tag{156}$$

where M_{ij} is the matrix obtained by removing row i and column j from A.

Remark. *The cofactor expansion can also be done along columns instead of rows:*

$$\boxed{\det A = a_{1j} \cdot C_{1j} + ... + a_{nj} \cdot C_{nj}} \tag{157}$$

for all $j \in \{1, ..., n\}$.

Let's do an example, by computing the determinant of the matrix:

$$A = \begin{bmatrix} 7 & 0 & 3 & -1 \\ 0 & 4 & 3 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

This matrix has a lot of zeroes, so it's perfect for cofactor expansion. To keep your work as light as possible, pick a row with a lot of zeroes. For example, cofactor expansion for the third row gives:

$$\det A = 2 \cdot (-1)^{3+1} \det \begin{bmatrix} 0 & 3 & -1 \\ 4 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix} + 1 \cdot (-1)^{3+4} \det \begin{bmatrix} 7 & 0 & 3 \\ 0 & 4 & 3 \\ 0 & 2 & 0 \end{bmatrix}$$

We can compute the determinants above by cofactor expansion for the first and third rows, respectively:

$$\det A = 2 \cdot (-1)^{3+1} \left(3 \cdot (-1)^{1+2} \det \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} + (-1) \cdot (-1)^{1+3} \det \begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} \right) +$$

$$+1 \cdot (-1)^{3+4} \cdot 2 \cdot (-1)^{3+2} \det \begin{bmatrix} 7 & 3 \\ 0 & 3 \end{bmatrix} = 2((-3) \cdot 4 + (-1) \cdot (-6)) + 2 \cdot 21 = 30$$

Just to do a sanity check, let's also compute the determinant of A by cofactor expansion along a column (say the second column):

$$\det A = 4 \cdot (-1)^{2+2} \det \begin{bmatrix} 7 & 3 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 2 \cdot (-1)^{2+4} \det \begin{bmatrix} 7 & 3 & -1 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Now let's compute the determinants above by cofactor expansion for the second and third column, respectively:

$$\det A = 4 \cdot (-1)^{2+2} \left(3 \cdot (-1)^{1+2} \det \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right) + 2 \cdot (-1)^{2+4} \left((-1) \cdot (-1)^{1+3} \det \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} + \right.$$

$$\left. + 1 \cdot (-1)^{3+3} \det \begin{bmatrix} 7 & 3 \\ 0 & 3 \end{bmatrix} \right) = 4 \cdot (-3) \cdot 2 + 2((-1) \cdot (-6) + 1 \cdot 21) = 30$$

Lecture 19 (*October 23*) *Reading: section 5.3*

Remember systems of linear equations like $A\boldsymbol{v} = \boldsymbol{b}$? We learned how to solve these through Gaussian elimination, but in the case when A is a square $n \times n$ matrix, there is another way called Cramer's rule. In this setup, you have as many equations as unknowns, so you (almost always) expect there to be a single solution. The phrase "almost always" should be interpreted to mean "when A is invertible", and in this case we have:

$$A\boldsymbol{v} = \boldsymbol{b} \qquad \text{has the unique solution} \qquad \boldsymbol{v} = A^{-1}\boldsymbol{b} \tag{158}$$

So the task has become to compute the inverse matrix A^{-1}. But if you look back at formula (155), the inverse matrix is almost staring back at you! Explicitly, consider the matrix:

$$X \quad \text{whose entries are} \quad x_{ij} = C_{ji}$$

and C_{ij} are the cofactors (156). Then formula (155) precisely says that:

$$\det A = a_{i1}x_{1i} + ... + a_{in}x_{ni}$$

for all $i \in \{1, ..., n\}$. Together with the fact that for all $i \neq j$:

$$0 = a_{i1}x_{1j} + ... + a_{in}x_{nj}$$

(proof: the right-hand side of the formula above is precisely the right-hand side of (155) for the matrix A' obtained from A by copying the i–row into of the j–th row; since A' has two equal rows, we have $\det A' = 0$) we conclude that:

$$\begin{bmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \det A \end{bmatrix} = AX$$

and therefore:

$$\boxed{A^{-1} = \frac{X}{\det A}} \tag{159}$$

In other words, the (i, j) entry of A^{-1} is given by:

$$\boxed{A^{-1}_{ij} = \frac{C_{ji}}{\det A}} \tag{160}$$

Note that you need to put the (j, i)–cofactor in the formula above, not the (i, j)–cofactor!
So now let's write a formula for the solution of the equation:

$$A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{161}$$

By (158), we must have:

$$v_i = A^{-1}_{i1} \cdot b_1 + ... + A^{-1}_{in} \cdot b_n$$

By formula (160), we therefore have:

$$v_i = \frac{C_{1i} \cdot b_1 + ... + C_{ni} \cdot b_n}{\det A}$$

If you stare the right-hand side of the formula above, you will notice that it's precisely the column cofactor expansion (157) of the matrix:

$$B_i \text{ obtained by replacing the } i\text{–th column of } A \text{ by } \boldsymbol{b} \tag{162}$$

Therefore, we get **Cramer's rule**, which says that the solution to the system (161) has:

$$\boxed{v_1 = \frac{\det B_1}{\det A}, ..., v_n = \frac{\det B_n}{\det A}} \tag{163}$$

This formula is particularly good when solving small systems of equations, like 3×3 ones:

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

By (163), the solution is:

$$v_1 = \frac{\det \begin{bmatrix} 0 & -2 & 0 \\ 4 & 0 & -3 \\ 1 & 0 & -5 \end{bmatrix}}{\det \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 2 & 0 & -5 \end{bmatrix}} \qquad v_2 = \frac{\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & -3 \\ 2 & 1 & -5 \end{bmatrix}}{\det \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 2 & 0 & -5 \end{bmatrix}} \qquad v_3 = \frac{\det \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 4 \\ 2 & 0 & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 2 & 0 & -5 \end{bmatrix}}$$

The 3×3 determinants above can be computed either by cofactor expansion (155), (157) (preferably along a row or column with many zeroes) or even by applying the brute force formula (152). In any case, we obtain:

$$v_1 = -17 \qquad v_2 = -\frac{17}{2} \qquad v_3 = -7$$

As a final applcation of determinants, let us discuss about cross products. Remember when I said that it does not make sense to multiply vectors and get a vector, bar a rather particular exception for three dimensional vectors? Well, here is that exception. If you have vectors:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

in three-dimensional space, their **cross product** is defined as the following vector in three dimensional space:

$$\boldsymbol{v} \times \boldsymbol{w} = \det \begin{bmatrix} \boldsymbol{i} & v_1 & w_1 \\ \boldsymbol{j} & v_2 & w_2 \\ \boldsymbol{k} & v_3 & w_3 \end{bmatrix} \tag{164}$$

where $\boldsymbol{i} = \boldsymbol{e}_1, \boldsymbol{j} = \boldsymbol{e}_2, \boldsymbol{k} = \boldsymbol{e}_3$ are the standard unit vectors on the axes of three-dimensional space. You can compute the determinant in (164) by cofactor expansion on the first column, and obtain:

$$\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{i} \cdot (v_2 w_3 - v_3 w_2) + \boldsymbol{j} \cdot (v_3 w_1 - v_1 w_3) + \boldsymbol{k} \cdot (v_1 w_2 - v_2 w_1) = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \tag{165}$$

The determinant definition of the cross-product has a few advantages: it is easy to memorize, and it manifestly explains why the cross product is 0 if the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are on the same line (becuse then columns 2 and 3 of the determinant (164) would be multiples of each other). Moreover, because switching two columns flips the sign of the determinant, we conclude that:

$$\boldsymbol{v} \times \boldsymbol{w} = -(\boldsymbol{w} \times \boldsymbol{v})$$

Moreover, if we have a third vector $\boldsymbol{u}$, then we could dot it with the cross product $\boldsymbol{v} \times \boldsymbol{w}$ and get a number. This number is actually quite nice, namely:

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \tag{166}$$

Geometrically, the determinant in the right-hand side of (166) is the volume of the parallelogram whose edges are given by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. If $\boldsymbol{u}$ lies in the plane spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$, then the volume of this parallelogram is 0. This is manifestly apparent from (166), since a matrix where one of the columns is a linear combination of the others has determinant 0.

Lecture 20 (*October 25)* *Reading: section 6.1*

Just like in the last few classes, we're sticking with square matrices, so let A be an $n \times n$ matrix. The goal is to make the matrix "as simple as possible". Some of the simplest square matrices out

there are **diagonal** matrices:

$$\text{diag}_{d_1,...,d_n} = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} \tag{167}$$

Indeed, the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ that corresponds to $\text{diag}_{d_1,...,d_n}$ simply scales along the unit vectors $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ of the coordinate axes (specifically, it scales by a factor of d_i along the i–th coordinate axis of $\mathbb{R}^n$). In particular, we have $\det(\text{diag}_{d_1,...,d_n}) = d_1...d_n$, which reflects the fact that our linear transformation scales volumes by a factor equal to $d_1...d_n$.

Definition 18. *A matrix is called **diagonalizable** if it is conjugate to a diagonal matrix, i.e.:*

$$A = V \cdot \text{diag}_{d_1,...,d_n} \cdot V^{-1} \tag{168}$$

for some invertible $n \times n$ matrix V, and some numbers $d_1, ..., d_n$.

The property (168) *means that the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ that corresponds to A scales by a factor of d_i in the direction of the vector $\boldsymbol{v}_i$, where $\boldsymbol{v}_i$ is the i–th column of V. Therefore, a diagonalizable matrix is diagonal in some basis of $\mathbb{R}^n$.*

Remark. *Almost all $n \times n$ matrices are diagonalizable, and next week we will see what one can say about those matrices which are not.*

Suppose you knew that the matrix A is diagonalizable, but you didn't know the specific numbers d_i and the matrix V. You could reconstruct them from the notion of eigenvalues and eigenvectors:

Definition 19. *Given a $n \times n$ matrix A, a non-zero vector $\boldsymbol{v} \in \mathbb{R}^n$ is called an **eigenvector** if:*

$$\boxed{A\boldsymbol{v} = \lambda \boldsymbol{v}} \tag{169}$$

*for some number λ called an **eigenvalue**.*

Indeed, $\boldsymbol{v}$ is an eigenvector of A with corresponding eigenvalue λ if the linear transformation corresponding to A scales by a factor of λ in the direction of the vector $\boldsymbol{v}$. This story is very closely connected with (168): if the matrix A is equal to $V\text{diag}_{d_1,...,d_n}V^{-1}$, then all the d_i's are eigenvalues of A and the corresponding eigenvectors are the columns of V.

So how many eigenvalues/eigenvectors are there? It's easy to see that if $\boldsymbol{v}$ is an eigenvector, then so is any multiple of $\boldsymbol{v}$, so a better question would be "how many linearly independent eigenvectors are there"? Well, from the equation (169) we obtain:

$$(A - \lambda I)\boldsymbol{v} = 0$$

Since $\boldsymbol{v}$ is to be non-zero, this implies that the matrix $A - \lambda I$ is to be singular (i.e. not invertible). Therefore, Fact 14 implies that:

$$\boxed{\det(A - \lambda I) = 0} \tag{170}$$

for any eigenvalue λ of A. Conversely, any number λ for which (170) holds is an eigenvalue of A (try to think of an argument). This leads us to the following:

Definition 20. *The **characteristic polynomial** of A is:*

$$p(\lambda) = \det(A - \lambda I) \tag{171}$$

as a function of the variable λ. The roots (i.e. those values λ for which $p(\lambda) = 0$) of the characteristic polynomial are the eigenvalues of A.

Although not very practical, you could in theory compute the characteristic polynomial by applying the big formula (153) for the determinant:

$$p(\lambda) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix} = \sum_{\{\sigma(1),\dots,\sigma(n)\}}^{\text{permutations}} (-1)^{\text{sgn } \sigma} a'_{1\sigma(1)} a'_{2\sigma(2)} \dots a'_{n\sigma(n)}$$

where $a'_{ij} = a_{ij}$ if $i \neq j$ and $a'_{ii} = a_{ii} - \lambda$. Since every summand in the right-hand side is a product of n numbers a'_{ij} for various i and j, no summand can contain more than n factors of λ. Moreover, the only summand which contains exactly n factors of λ is $a'_{11} a'_{22} \dots a'_{nn} = (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$. Therefore, the characteristic polynomial has degree n in λ and its top degree coefficient is $(-1)^n$:

$$p(\lambda) = (-1)^n \lambda^n + \alpha_{n-1} \lambda^{n-1} \dots + \alpha_1 \lambda + \alpha_0$$

The coefficients $\alpha_{n-1}, \dots, \alpha_0$ depend on the entries of the matrix A. For example:

$$\alpha_{n-1} = (-1)^{n-1} \cdot \text{tr } A \tag{172}$$

where:

$$\boxed{\text{tr } A = a_{11} + a_{22} + \dots + a_{n-1,n-1} + a_{nn}} \tag{173}$$

is called the **trace** of the matrix A. Clearly:

$$\alpha_0 = \det A \tag{174}$$

is just the determinant of A (this is just saying that the value of the characteristic polynomial at $\lambda = 0$ is the determinant of A, which is obvious given the definition (171)).

Fact 16. *The sum of the eigenvalues of a matrix A is equal to the trace of A.*

The product of the eigenvalues of a matrix A is the determinant of A.

As an exercise, compute the eigenvalues of the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \tag{175}$$

First let us form the characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 5 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda) - 5 \cdot 1 = \lambda^2 - 4\lambda - 1$$

The eigenvalues of A are the roots of this polynomial, which you can get by the quadratic formula:

$$\lambda = \frac{4 \pm \sqrt{4^2+4}}{2} = 2 \pm \sqrt{5} \qquad \Rightarrow \qquad \begin{cases} \lambda_1 = 2 + \sqrt{5} \\ \lambda_2 = 2 - \sqrt{5} \end{cases}$$

In general, because the eigenvalues of a 2×2 matrix are the solutions of a quadratic equation, you should expect to encounter square roots in their formulas. As we have seen in the example above, in the 2×2 case, one can deduce the formula for the characteristic polynomial from the facts that:

- it is quadratic and its leading coefficient is 1
- its linear term is minus the trace of the matrix (172)
- its constant term is the determinant of a matrix (174)

So the characteristic polynomial of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:

$$p(\lambda) = \lambda^2 - \lambda(a+d) + (ad - bc) \tag{176}$$

For matrices of size 3×3 and higher, the characteristic polynomial is of order 3 and higher, so it is a bit more complicated to write it down and a lot more complicated to find its roots.

Remark. *In general, it is* ***NOT*** *true that the eigenvalues of $A+B$ are the sums of the eigenvalues of A and B, nor are the eigenvalues of AB the products of the eigenvalues of A and B.*

Now suppose that you know λ is an eigenvalue of a square matrix A; how do you find an eigenvector $\boldsymbol{v}$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$? You already have the tools for this: just recast the eigenvector relation as $(A - \lambda I)\boldsymbol{v} = 0$, so all you need to do is to find $\boldsymbol{v} \neq 0$ in the nullspace of the singular matrix $A - \lambda I$. For example, in the case of the matrix (175), let us look for an eigenvector $\boldsymbol{v}$ corresponding to the eigenvalue $\lambda = 2 + \sqrt{5}$. To this end, let us construct the matrix:

$$A - \lambda I = \begin{bmatrix} -\sqrt{5} & 1 \\ 5 & -\sqrt{5} \end{bmatrix}$$

and we will simply pick a vector $\boldsymbol{v} \in N(A - \lambda I)$. Remember that, in order to compute the nullspace of a matrix, the way to go is to put it in reduced row echelon form:

$$\begin{bmatrix} -\sqrt{5} & 1 \\ 5 & -\sqrt{5} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -\sqrt{5} & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -\frac{1}{\sqrt{5}} \\ 0 & 0 \end{bmatrix}$$

Therefore, an eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ can be obtained by solving the equation:

$$\begin{bmatrix} 1 & -\frac{1}{\sqrt{5}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \qquad \Leftrightarrow \qquad v_1 - \frac{v_2}{\sqrt{5}} = 0$$

So one particular choice of eigenvector can be obtained by setting the free variable v_2 equal to 1, and solving for the pivot variable from the equation above: $v_1 = \frac{1}{\sqrt{5}}$. Therefore, an eigenvector of the matrix A corresponding to the eigenvalue $\lambda = 2 + \sqrt{5}$ is:

$$\boldsymbol{v} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 1 \end{bmatrix}$$

Any multiple of $\boldsymbol{v}$ would be an equally good eigenvector.

Lecture 21 (*October 28*) *Reading: section 6.2*

In the previous class, we saw how to compute the eigenvalues and eigenvectors of an $n \times n$ matrix A. Specifically, the eigenvalues are the roots of the characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = 0$$

Since the characteristic polynomial has degree n, it has at most n roots, therefore there are at most n eigenvalues. But there could be fewer than n eigenvalues, for example the matrix:

$$A = \begin{bmatrix} 5 & -3 \\ 0 & 5 \end{bmatrix}$$

has characteristic polynomial

$$p(\lambda) = (\lambda - 5)^2$$

which has $\lambda = 5$ as a double root. In this case, we say that $\lambda = 5$ has **algebraic multiplicity** 2, or that the matrix A has repeated (or equal) eigenvalues. In other words, we still think of A has having two eigenvalues, but they are both equal to 5. With this in mind, we have:

Fact 17. *An $n \times n$ matrix has exactly n eigenvalues, although some could be repeated, and some could be complex numbers (more on the latter later).*

So what can we say about eigenvectors? First of all, eigenvectors corresponding to different eigenvalues are always linearly independent. For example, if:

$$A\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1 \qquad \text{and} \qquad A\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2$$

for $\lambda_1 \neq \lambda_2$, then there can be no linear relation between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (argument: if there were such a relation, then we would have $\boldsymbol{v}_1 = c\boldsymbol{v}_2$ for some scalar c, and applying A would give us $\lambda_1 \boldsymbol{v}_1 = A\boldsymbol{v}_1 = cA\boldsymbol{v}_2 = c\lambda_2 \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1$, which would force $\lambda_1 = \lambda_2$). So the best case is when the n eigenvalues of the matrix are all distinct numbers, because then there would exist n linearly independent eigenvectors, one for each eigenvalue. These eigenvectors would form a basis of $\mathbb{R}^n$.

Fact 18. *An $n \times n$ matrix A with n distinct eigenvalues $\lambda_1, ..., \lambda_n$ is diagonalizable. Specifically:*

$$A = V \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} V^{-1} \tag{177}$$

where V is the matrix whose columns are $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$, the eigenvectors corresponding to $\lambda_1, ..., \lambda_n$.

A couple of remarks: if the eigenvalues are all distinct, then the eigenvectors are really only defined up to constant multiple. So the matrix V in (177) is not unique, but it depends on a choice of constant multiples of the eigenvectors. Secondly, the fact that the eigenvectors are linearly independent (which was discussed right before Fact 18) is crucial, because otherwise the matrix V would have rank $< n$ and would not be invertible and the right-hand side of formula (177) would not make sense.

In practice, how to diagonalize (i.e. to write as in (177)) a matrix A? As explained in Fact 18, the way to do so is to find the eigenvalues and eigenvectors of A. Say we're looking at the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{178}$$

To find its eigenvalues, construct the characteristic polynomial:

$$p(\lambda) = \lambda^2 - \lambda \cdot \text{tr } A + \det A = \lambda^2 - \lambda - 1$$

(here we're applying the shortcut formula (176), which only holds in the 2×2 case). Its roots are:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Let us now compute eigenvectors corresponding to these two eigenvalues:

$$A\boldsymbol{v}_1 = \frac{1+\sqrt{5}}{2}\boldsymbol{v}_1 \quad \Rightarrow \quad \left(A - \frac{1+\sqrt{5}}{2}I\right)\boldsymbol{v}_1 = 0 \quad \Rightarrow \quad \boldsymbol{v}_1 \in N\left(A - \frac{1+\sqrt{5}}{2}I\right)$$
$$A\boldsymbol{v}_2 = \frac{1-\sqrt{5}}{2}\boldsymbol{v}_2 \quad \Rightarrow \quad \left(A - \frac{1-\sqrt{5}}{2}I\right)\boldsymbol{v}_2 = 0 \quad \Rightarrow \quad \boldsymbol{v}_2 \in N\left(A - \frac{1-\sqrt{5}}{2}I\right)$$

To work out the nullspace of $A - \frac{1+\sqrt{5}}{2}I$, put it in row echelon form:

$$A - \frac{1+\sqrt{5}}{2}I = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix}$$

So the eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ needs to satisfy:

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \Leftrightarrow \quad y + \frac{1-\sqrt{5}}{2}x = 0$$

and therefore we may take $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix}$. Similarly, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{bmatrix}$, so we have:

$$A = V \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} V^{-1} \qquad \text{where} \qquad V = \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \tag{179}$$

So we've done all this work, but you might ask: what are applications of diagonalization? Well, here's a classic one. Let's say you want to obtain an explicit formula for the Lucas numbers (close cousins of the Fibonacci numbers considered in the textbook), which are defined by the conditions:

$$L_0 = 2, \qquad L_1 = 1, \qquad \text{and the recursive relation} \qquad L_{n+1} = L_n + L_{n-1} \tag{180}$$

for all $n \geq 1$. One way would be to consider the vectors:

$$\boldsymbol{a}_n = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$$

and to observe that the conditions (180) can be interpreted as the matrix identities:

$$\boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{a}_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_{n-1}$$

for all $n \geq 2$. You can iterate the latter formula and obtain:

$$\boldsymbol{a}_n = A\boldsymbol{a}_{n-1} = A^2\boldsymbol{a}_{n-2} = ... = A^n\boldsymbol{a}_0 = A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{181}$$

where A is the matrix (178). So computing a formula for the Lucas numbers boils down to having an effective method for explicitly computing the matrix powers A^n for all natural numbers n. This is tricky for a general matrix A, but quite straightforward if you kow how to diagonalize A. Specifically, because A can be written in the form (179), we have:

$$A^2 = V \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \underbrace{V^{-1}V}_{I} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} V^{-1} = V \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^2 & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^2 \end{bmatrix} V^{-1}$$

$$A^3 = V \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \underbrace{V^{-1}V}_{I} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \underbrace{V^{-1}V}_{I} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} V^{-1} = V \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^3 & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^3 \end{bmatrix} V^{-1}$$

...

$$A^nV = V \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} V^{-1} \qquad \text{for all integers } n$$

We could plug these formulas in (181) and obtain:

$$\begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = \boldsymbol{a}_n = V \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} V^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \frac{1}{-\sqrt{5}} \begin{bmatrix} \frac{-1-\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

I'll let you have all the fun in multiplying the matrices on the bottom row above. When you do so, you will finally obtain the formula for the Lucas numbers:

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{182}$$

The example we have just seen generalizes to the following principle:

$$\text{if} \quad A = V \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} V^{-1} \quad \text{then} \quad A^k = V \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} V^{-1} \tag{183}$$

This is computationally very important, because if you want to compute the k–th power of an $n \times n$ matrix efficiently for some really large number k, it is faster to just diagonalize A and apply formula (183). Also, this formula allows us to describe the behavior of A^k as $k \to \infty$: its growth rate is dominated by the size of the k–th power of the largest eigenvalue of A.

Remark. *The Lucas numbers L_n grow like $\left(\frac{1+\sqrt{5}}{2}\right)^n$ as $n \to \infty$. This makes sense, since the other eigenvalue $\frac{1-\sqrt{5}}{2}$ is contained between -1 and 1, and so its n–th powers will converge to 0 as $n \to \infty$.*

Geometrically, suppose you want to compute the value of A^k applied to any vector $\boldsymbol{v}$? Just decompose this vector as a linear combination of the eigenvectors:

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n \quad \Rightarrow \quad A^k\boldsymbol{v} = c_1A^k\boldsymbol{v}_1 + ... + c_nA^k\boldsymbol{v}_n$$

and so we obtain the formula:

$$\boxed{A^k\boldsymbol{v} = c_1\lambda_1^k\boldsymbol{v}_1 + ... + c_n\lambda_n^k\boldsymbol{v}_n} \tag{184}$$

We conclude that the powers of A scale by various powers of the eigenvalues λ_1,...,λ_n in the direction of the eigenvectors $\boldsymbol{v}_1$,...,$\boldsymbol{v}_n$, and this gives an effective way to compute A^k times any vector.

Finally, let us mention a common-sense, but important thing: **the eigenvalues of a diagonal matrix are the entries on the diagonal**. This is because the roots of the polynomial:

$$\det\left(\begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} - \lambda \cdot I\right) = \det\begin{bmatrix} d_1-\lambda & 0 & \dots & 0 \\ 0 & d_2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n-\lambda \end{bmatrix} = (d_1-\lambda)...(d_n-\lambda)$$

are precisely $d_1, ..., d_n$. Applying this to (177), we see that the matrices A and $\text{diag}_{\lambda_1,...,\lambda_n}$ have the same eigenvalues. This is true for any pair of conjugate matrices:

$$\textbf{if } A = VBV^{-1} \textbf{ then } A \textbf{ and } B \tag{185}$$
$$\textbf{have the same eigenvalues and characteristic polynomial}$$

for any invertible matrix V. However, A and B will in general have different eigenvectors, and these two bases of eigenvectors will be connected by the change of basis matrix V in formula (185).

Lecture 22 (*October 30*) *Reading: section 8.3*

The previous class was all about the situation when an $n \times n$ matrix A has distinct eigenvalues. But what if the eigenvalues are repeated, i.e. what if the characteristic polynomial takes the form:

$$p(\lambda) = (d_1 - \lambda)^{r_1}(d_2 - \lambda)^{r_2}...(d_s - \lambda)^{r_s} \tag{186}$$

for certain distinct numbers $d_1, ..., d_s$, and for certain powers $r_1, ..., r_s$? We think of the eigenvalues of A as being the following sequence of numbers:

$$\underbrace{d_1, ..., d_1}_{r_1 \text{ times}}, \underbrace{d_2, ..., d_2}_{r_2 \text{ times}}, ..., \underbrace{d_s, ..., d_s}_{r_s \text{ times}} \tag{187}$$

The number r_i is called the **algebraic multiplicity** of the eigenvalue d_i, and it represents the number of times it appears among the roots of the characteristic polynomial. Because the degree of the characteristic polynomial is n, the sum of the algebraic multiplicities of all eigenvalues is:

$$r_1 + ... + r_s = n \tag{188}$$

What about eigenvectors for each eigenvalue d_i? The set of eigenvectors for any given eigenvalue is a subspace of $\mathbb{R}^n$, namely:

$$\Big\{\text{eigenvectors of } A \text{ corresponding to eigenvalue } \lambda\Big\} = N(A - \lambda I)$$

The **geometric multiplicity** of any eigenvalue d_i is defined as the dimension of the corresponding space of eigenvectors, namely $\dim N(A - d_i I)$. It never exceeds the algebraic multiplicity:

$$\textbf{geometric multiplicity} \leq \textbf{algebraic multiplicity} \tag{189}$$

For example, consider the matrices below:

$$A = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix} \tag{190}$$

Both of them have characteristic polynomial $(\lambda - d)^2$, so d is the only eigenvalue, with algebraic multiplicity 2. But the geometric multiplicity differs among the two matrices. In the case of the matrix A, any vector is an eigenvector because $A\boldsymbol{v} = d\boldsymbol{v}$ for all $\boldsymbol{v}$, thus implying that the eigenvalue d has geometric multiplicity 2. But for the matrix B, we have that:

$$\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ is an eigenvector} \quad \Leftrightarrow \quad \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = d \begin{bmatrix} x \\ y \end{bmatrix} \quad \Leftrightarrow \quad y = 0$$

which implies that the subspace of eigenvectors is one-dimensional, hence the eigenvalue d has geometric multiplicity 1. This is strictly smaller than its algebraic multiplicity, which is 2.

Fact 19. *A matrix is diagonalizable if and only if all of its eigenvalues have geometric multiplicity equal to their algebraic multiplicity.*

Indeed, if the geometric multiplicities of all eigenvalues are equal to the algebraic multiplicities, then (188) implies that the sum of all geometric multiplicities is equal to n. This implies that the subspaces of eigenvectors have dimensions which add up to n, hence we can pick a basis of $\mathbb{R}^n$ consisting only of eigenvectors. This allows us to define the invertible matrix V whose columns are the n chosen eigenvectors, and formula (177) holds.

But what can we say when some of the eigenvalues have geometric multiplicities strictly smaller than their algebraic ones? In this case, the matrix A is no longer conjugate to a diagonal matrix, as in (177). Instead, it will be conjugate to a matrix in **Jordan normal form**:

$$A = V \left[\begin{array}{c|c|c|c} J_1 & 0 & \dots & 0 \\ \hline 0 & J_2 & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & J_t \end{array}\right] V^{-1} \tag{191}$$

where each J_i is a **Jordan block**, i.e. a matrix of the form:

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{bmatrix} \tag{192}$$

where λ is among the eigenvalues of A. The total number of d_i's that appear on the diagonal of the block matrix in (191) is equal to r_i, the algebraic multiplicity of the eigenvalue d_i.

There are two things one needs to know in order to set up the Jordan normal form (191): the specific sizes of the Jordan blocks $J_1, ..., J_t$ and the matrix V. There is a general prescription for figuring out the sizes of the Jordan blocks, but it goes beyond the scope of our course. As for the matrix V, we know from the diagonalizable case that its columns have something to do with the nullspaces $N(A - d_1 I), ..., N(A - d_s I)$. However, in the situation at hand (when the geometric multiplicities are smaller than the algebraic multiplicities), the dimensions of these nullspaces do not add up to n, so they will not contain n linearly independent vectors. So instead, look at:

$$N\Big((A - d_1 I)^{r_1}\Big), ..., N\Big((A - d_s I)^{r_s}\Big) \tag{193}$$

(where $r_1, ..., r_s$ are the algebraic multiplicities of the eigenvalues) and we claim that these subspaces are both linearly independent, and span the entire $\mathbb{R}^n$. Then the subspaces (193) will contain n linearly independent vectors, and these will be the columns of the matrix V of (191).

Let's present this in more detail in the case when the algebraic multiplicities are all either 1 or 2. Consider the matrix:

$$A = \begin{bmatrix} 10 & -3 & 7 \\ 27 & -10 & 18 \\ 5 & -3 & 3 \end{bmatrix}$$

and we want to compute its Jordan normal form: i.e. a matrix V and Jordan blocks $J_1, ..., J_t$ such

that formula (191) holds. First of all, let's compute the characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 10-\lambda & -3 & 7 \\ 27 & -10-\lambda & 18 \\ 5 & -3 & 3-\lambda \end{bmatrix}$$

Let us compute the determinant by cofactor expansion along the last row:

$$p(\lambda) = 5 \det \begin{bmatrix} -3 & 7 \\ -10-\lambda & 18 \end{bmatrix} - (-3) \det \begin{bmatrix} 10-\lambda & 7 \\ 27 & 18 \end{bmatrix} + (3-\lambda) \det \begin{bmatrix} 10-\lambda & -3 \\ 27 & -10-\lambda \end{bmatrix} =$$

$$= 5(-54+70+7\lambda) + 3(180-18\lambda-189) + (3-\lambda)(\lambda^2 - 100 + 81) = -\lambda^3 + 3\lambda^2 - 4$$

It's not easy to find the roots of a general cubic polynomial. But when you're out of ideas, just try plugging in some simple numbers for λ to see if $p(\lambda) = 0$. For example, trying $\lambda = -1$ shows that $p(-1) = 0$, which implies that $p(\lambda)$ is divisible by $\lambda + 1$. Then polynomial long division shows that:

$$p(\lambda) = (\lambda+1)(-\lambda^2 + 4\lambda - 4) = -(\lambda+1)(\lambda-2)^2$$

So the eigenvalues are $d_1 = -1$ with algebraic multiplicity one, and $d_2 = 2$ with algebraic multiplicity two. Now for the subspaces of eigenvectors. We have:

$$A + I = \begin{bmatrix} 11 & -3 & 7 \\ 27 & -9 & 18 \\ 5 & -3 & 4 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{hence} \quad N(A+I) = N\left(\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \right)$$

The matrix on the right has pivot columns 1 and 2, and free column 3, so its null-space is one dimensional and generated by the vector:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{such that} \quad \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \qquad \Rightarrow x = -\frac{1}{2} \text{ and } y = \frac{1}{2}$$

Therefore, the eigenvalue $d_1 = -1$ also has geometric multiplicity one, and an eigenvector is:

$$\boldsymbol{v}_1 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

As for the other eigenvalue, we have:

$$A - 2I = \begin{bmatrix} 8 & -3 & 7 \\ 27 & -12 & 18 \\ 5 & -3 & 1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{hence} \quad N(A-2I) = N\left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

The matrix on the right has pivot columns 1 and 2, and free column 3, so its null-space is one dimensional and generated by the vector:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{such that} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \qquad \Rightarrow x = -2 \text{ and } y = -3$$

Therefore, the eigenvalue $d_2 = 2$ has geometric multiplicity one, and an eigenvector is:

$$\boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

So we are in the situation when geometric multiplicity is strictly smaller than algebraic multiplicity, hence the subspaces of eigenvectors are not enough to give rise to a basis of $\mathbb{R}^3$. In this case, (193) tells us to replace the nullspace of $A - 2I$ by the nullspace of $(A - 2I)^2$, namely:

$$(A-2I)^2 = \begin{bmatrix} 18 & -9 & 9 \\ -18 & 9 & -9 \\ -36 & 18 & -18 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{hence} \quad N(A+I) = N\left(\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

The matrix on the right has pivot column 1 and free columns 2 and 3, so its nullspace is two dimensional and consists of vectors:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{such that} \quad 2x - y + z = 0$$

One of these vectors is $\boldsymbol{v}_2$, but we can take any other such vector, let's say $\boldsymbol{v}_3 = \begin{bmatrix} a \\ 2a \\ 0 \end{bmatrix}$ (there will be a benefit in making the last entry 0, as we will soon see) for any a. Therefore, the matrix:

$$V = \left[\begin{array}{c|c|c} & & \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ & & \end{array} \right] = \begin{bmatrix} -\frac{1}{2} & -2 & a \\ \frac{1}{2} & -3 & 2a \\ 1 & 1 & 0 \end{bmatrix}$$

is invertible. The constant $a \neq 0$ is yours to choose freely, but if you want to have the equality:

$$A = V \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 0 & 2 \end{array} \right] V^{-1}$$

then you need to choose $a = -1$ (if you were to use any other value for a, then you would just need to replace the entry 1 in the 2×2 Jordan block above by the entry $-a$, which is not too bad).

Remark. *As I mentioned, I did not give you a complete recipe for how to set up the Jordan blocks in* (191)*. But as you were able to see from the example above, if the Jordan blocks corresponding to an eigenvalue λ are located on columns $i, ..., j$, then the columns $i, ..., j$ of the matrix V need to be filled with a basis of $(N(A - \lambda I)^r)$, where r is the algebraic multiplicity of the eigenvalue λ.*

Lecture 23 (*November 4*) *Reading: section 6.3*

Diagonalization and Jordan normal forms can be applied in solving systems of ordinary differential equations. We will give a brief introduction to this topic here, but you can learn about it more systematically in 18.03. A **differential equation** is any equality of the form:

$$\dot{u}(t) = \text{any expression involving } u(t) \tag{194}$$

where $u(t)$ is a function of a single real variable (think of t as time) and $\dot{u} = u'$ is its derivative. The point of (194) is to solve for the function $u(t)$. As you can imagine, the problem is hard to solve in general, especially if the expression in the right-hand side of (194) is a complicated expression of $u(t)$. But we will focus on the case of **linear differential equations**, which are of the form:

$$\dot{u}(t) = \lambda u(t) \tag{195}$$

for a fixed number λ. These are now easy to solve. In fact, one can even guess the fact that:

$$u(t) = e^{\lambda t} c \tag{196}$$

is a solution of (195) for any constant c. Moreover, there's a theorem (the existence and uniqueness of solutions of ordinary differential equations) which states that there is a unique solution to (195) once the "initial value" $c = u(0)$ is prescribed. Therefore, the collection of functions (196) comprises of all solutions to the equation (195), and we have solved the latter equation completely.

The equation (195) is called **first order**, because only the first derivative of u appears in the equation. But one can consider instead the more general case of an n–th order linear differential equation, which will involve all derivatives of u up to the n–th one:

$$u^{(n)}(t) = a_{n-1} u^{(n-1)}(t) + ... + a_2 u^{(2)}(t) + a_1 \dot{u}(t) + a_0 u(t) \tag{197}$$

where $u^{(n)}(t)$ denotes the n–th derivative of u, i.e. u differentiated n times. Equation (197) is not so straightforward to solve anymore; in particular, it's not easy to guess a solution. However, there is a process of replacing a single n–th order linear differential equation such as (197) by a **system** of n first order linear differential equations. The way to do so is to introduce some more notation:

$$u_1(t) = u(t), \quad u_2(t) = \dot{u}(t), \quad u_3(t) = u^{(2)}(t), \quad \dots \quad u_n(t) = u^{(n-1)}(t) \tag{198}$$

Then the equation (197) becomes:

$$\dot{u}_n(t) = a_{n-1} u_n(t) + ... + a_1 u_2(t) + a_0 u_1(t)$$

to which we must add the common-sense properties (immediate consequences of (198)):

$$\dot{u}_1(t) = u_2(t), \qquad \dot{u}_2(t) = u_3(t), \qquad ... \qquad \dot{u}_{n-1}(t) = u_n(t)$$

With this in mind, solving the differential equation (197) is equivalent to solving the system:

$$\begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \vdots \\ \dot{u}_{n-1}(t) \\ \dot{u}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n-1}(t) \\ u_n(t) \end{bmatrix} \tag{199}$$

This is now a particular case of a **system of ordinary linear differential equations**:

$$\boxed{\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t)} \qquad \text{where} \qquad \boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} \tag{200}$$

is a vector of unknown functions that we wish to solve for, and A is an $n \times n$ matrix whose entries do not depend on t. The point of this lecture is to give a prescription for how to solve (200). First of all, let's deal with the case when A is a diagonal matrix. Then we have:

$$\begin{bmatrix} \dot{u}_1(t) \\ \vdots \\ \dot{u}_n(t) \end{bmatrix} = \begin{bmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_n \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} \dot{u}_1(t) = d_1 u_1(t) \\ \dots \\ \dot{u}_n(t) = d_n u_n(t) \end{cases} \tag{201}$$

This is just a collection of equations (195), so we know that the complete solution is:

$$\begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} e^{d_1 t} c_1 \\ \vdots \\ e^{d_n t} c_n \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{u}(t) = \begin{bmatrix} e^{d_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{d_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \tag{202}$$

The vector of constants $\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is equal to $\boldsymbol{u}(0)$, which is the initial condition.

Now assume we are solving the system of equations (200) for a diagonalizable matrix A (as we have seen in the past weeks, almost all matrices are diagonalizable). So we assume:

$$A = V \begin{bmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_n \end{bmatrix} V^{-1}$$

for an invertible matrix V. Then formula (200) becomes:

$$\dot{\boldsymbol{u}}(t) = V \begin{bmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_n \end{bmatrix} V^{-1} \boldsymbol{u}(t) \quad \Rightarrow \quad V^{-1} \dot{\boldsymbol{u}}(t) = \begin{bmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_n \end{bmatrix} V^{-1} \boldsymbol{u}(t)$$

If we write $\boldsymbol{w}(t) = V^{-1}\boldsymbol{u}(t)$, then the formula above reads precisely:

$$\dot{\boldsymbol{w}}(t) = \begin{bmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_n \end{bmatrix} \boldsymbol{w}(t)$$

This is precisely the system of equations (201), which has the solution (202):

$$\boldsymbol{w}(t) = \begin{bmatrix} e^{d_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{d_n t} \end{bmatrix} \boldsymbol{a}$$

for a vector of constants $\boldsymbol{a}$. Since the solution to the equation (200) is $\boldsymbol{u}(t) = V\boldsymbol{w}(t)$, we have:

$$\boldsymbol{u}(t) = V \begin{bmatrix} e^{d_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{d_n t} \end{bmatrix} V^{-1} \boldsymbol{c} \tag{203}$$

where $\boldsymbol{c} = V\boldsymbol{a} = \boldsymbol{u}(0)$. Let us summarize the discussion above by saying:

If $A = VDV^{-1}$, then the solutions to $\boxed{\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t)}$ and $\boxed{\dot{\boldsymbol{w}}(t) = D\boldsymbol{w}(t)}$ are related by $\boldsymbol{u}(t) = V\boldsymbol{w}(t)$. So solving the latter equation is equivalent to solving the former. (204)

In practice, the matrix D will be simple (e.g. diagonal) and so the latter equation will be easy to solve. This principle allows one to solve the former equation for a general matrix A.

Remark. *There's an alternative way to think about solving systems of differential equations. Basically, if you were to paraphrase the solution to* (195)*, then the solution to* (200) *should be:*

$$\boldsymbol{u}(t) = e^{At}\boldsymbol{c} \tag{205}$$

where the vector of constants $\boldsymbol{c}$ *is equal to* $\boldsymbol{u}(0)$*. Formula* (205) *is correct, but to make it precise, one must explain what is meant by* ***matrix exponential****. One way to define it is via Taylor series:*

$$e^{At} = 1 + At + \frac{A^2t^2}{2} + ... + \frac{A^nt^n}{n!} + ...$$

which is a practical way to define it for special matrices A *(such as upper/lower triangular matrices with 0 on the diagonal). Moeover, if* $A = VDV^{-1}$ *with* D *a diagonal matrix, then* (183) *implies:*

$$A^k = VD^kV^{-1} \quad \Rightarrow \quad e^{At} = Ve^{Dt}V^{-1}$$

With this in mind, the solution (205) *is precisely the same as* (203)*.*

As an example, let us solve the system of differential equations:

$$\begin{cases} \dot{u}_1(t) = 11u_1(t) - 6u_2(t) \\ \dot{u}_2(t) = 18u_1(t) - 10u_2(t) \end{cases}$$

Let us combine the unknown functions into the vector $\boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and convert the system into:

$$\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t) \quad \text{where} \quad A = \begin{bmatrix} 11 & -6 \\ 18 & -10 \end{bmatrix} \tag{206}$$

The second step is to diagonalize A, which starts with computing its eigenvalues:

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 11 - \lambda & -6 \\ 18 & -10 - \lambda \end{bmatrix} = \lambda^2 - \lambda - 2$$

has roots equal to 2 and -1. You may compute eigenvectors for each of these eigenvalues:

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in N(A - 2I) \qquad \text{and} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in N(A + I)$$

Then Fact 18 implies that:

$$A = V \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} V^{-1} \quad \text{where} \quad V = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

So the principle (204) allows us to convert the system (206) into the system:

$$\dot{\boldsymbol{w}}(t) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{w}(t)$$

where $\boldsymbol{w}(t) = V^{-1}\boldsymbol{u}(t)$. The latter system has solution:

$$\boldsymbol{w}(t) = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \boldsymbol{a}$$

which implies that the system (206) has solution:

$$\boldsymbol{u}(t) = V \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \boldsymbol{a} = \begin{bmatrix} 2a_1 e^{2t} + a_2 e^{-t} \\ 3a_1 e^{2t} + 2a_2 e^{-t} \end{bmatrix}$$

for certain constants a_1 and a_2, which one can solve for from the initial condition $\boldsymbol{u}(0)$.

Lecture 24 (*November 6*) *Reading: section 9.1*

Consider the problem of diagonalizing the matrix:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{207}$$

As we have seen in Fact 18, the recipe for doing so is to find the eigenvalues and eigenvectors of A first. Now the eigenvalues are the roots of the characteristic polynomial:

$$p(\lambda) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 \tag{208}$$

You may be tempted to believe that this polynomial has no roots, because after all, no real number has the property that its square is -1. And your guess would seem to be justified: the linear transformation corresponding to the matrix (207) is rotation by 90 degrees of the plane. If the matrix had an eigenvector, then rotation by 90 degrees would have to scale along the direction of that eigenvector, which does not seem to make any sense geometrically.

However, in math, a paradox is rarely the end of the story, but more often it is the catalyst for new discoveries. In the example at hand, the way to fix the fact that the polynomial (208) has no solutions is to add these solutions by hand. What this means is that we are no longer satisfied with real numbers, and instead we want to consider the **imaginary number**:

$$i \quad \text{defined such that} \quad \boxed{i^2 = -1} \tag{209}$$

which is often abbreviated as saying $i = \sqrt{-1}$. You should read this as saying that i is defined to be the square root of -1, and it certainly is not a real number. But it is a very useful tool for defining **complex numbers**, which are by definition all expressions of the form:

$$z = a + bi \tag{210}$$

for any real numbers a and b. You may ask: why don't we also throw in various powers of i into the expression (210) while we're at it? The reason is that doing so would be redundant, since the powers of i can themselves be expressed in the form (210):

$$i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 \qquad i^5 = i \qquad i^6 = -1 \qquad ... \tag{211}$$

Remark. *I hope you won't be put off by the feeling that complex numbers are "contrived". In the past, people have argued for hundreds of years whether negative integers make any sense, and ultimately we accepted them. And if you think about it, what is the meaning of "-2" other than "the thing to which you add 2 to get 0" and what is the meaning of "$\sqrt{3}$" other than "the thing which you square to get 3". Similarly, "i" is defined as "the thing which you square to get -1".*

Complex numbers can be added and multiplied using the usual algebra operations:

$$(a+ib)+(c+id)=(a+c)+i(b+d) \tag{212}$$

$$(a+ib)(c+id)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i \tag{213}$$

so the sum and product of complex numbers is also a complex number. But complex numbers have other kinds of operations associated to them. If $z=a+bi$ is a complex number, then:

$$\text{Re } z=a \text{ is called the real part of } z \tag{214}$$

$$\text{Im } z=b \text{ is called the imaginary part of } z \tag{215}$$

$$\bar{z}=a-bi \text{ is called the conjugate of } z \tag{216}$$

Therefore, the conjugate of a complex number is that complex number with the same real part, but the opposite imaginary part. Moreover, define the **absolute value** of a complex number as:

$$|z|=\sqrt{a^2+b^2} \tag{217}$$

If this looks familiar, then it should! Since a complex number (210) is just a pair of real numbers, it encodes the same information as a point in the plane, or a vector in $\mathbb{R}^2$. Then the absolute value of the complex number is just the length of the corresponding vector. Let us prove that:

$$\boxed{z\bar{z}=|z|^2} \tag{218}$$

Indeed, this is because:

$$(a+bi)(a-bi)=a^2-abi+abi-b^2i^2=a^2+b^2$$

Formula (218) is important because it allows us to write:

$$\frac{1}{z}=\frac{\bar{z}}{|z|^2} \qquad \text{i.e.} \qquad \frac{1}{a+bi}=\frac{a-bi}{a^2+b^2} \tag{219}$$

and this is great because it allows us to divide complex numbers:

$$\frac{2+i}{4+3i}=\frac{(2+i)(4-3i)}{4^2+3^2}=\frac{8-6i+4i-3i^2}{25}=\frac{11}{25}-\frac{2}{25}i$$

We now have the tools to diagonalize the matrix (207). Its characteristic polynomial has two roots, which are complex numbers:

$$p(\lambda)=\lambda^2+1=(\lambda-i)(\lambda+i) \qquad \text{has roots} \qquad \begin{cases}\lambda_1=i\\ \lambda_2=-i\end{cases}$$

so the eigenvalues are i and $-i$. Now we need to find eigenvectors corresponding to these eigenvalues:

$$\begin{array}{lllll} A\boldsymbol{v}_1 = i\boldsymbol{v}_1 & \Rightarrow & (A - iI)\boldsymbol{v}_1 = 0 & \Rightarrow & \boldsymbol{v}_1 \in N(A - iI) \\ A\boldsymbol{v}_2 = -i\boldsymbol{v}_2 & \Rightarrow & (A + iI)\boldsymbol{v}_2 = 0 & \Rightarrow & \boldsymbol{v}_2 \in N(A + iI) \end{array}$$

To compute the nullspaces in question, we perform our good friend Gaussian elimination:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \overset{\text{REF}}{\rightsquigarrow} \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$$

just by adding $(-i)$ times the first row to the second row. Therefore, $\boldsymbol{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ lies in $N(A - iI)$ if:

$$\begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \Rightarrow \quad -ix - y = 0$$

so a choice of eigenvector for the eigenvalue i is $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Similarly, one finds that a choice of eigenvector for the eigenvalue $-i$ is $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Therefore, Fact 18 implies that:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = V \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} V^{-1} \qquad \text{where} \qquad V = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

and this concludes the diagonalization of the matrix A of (207).

Any quadratic polynomial has two complex roots, specifically:

$$ax^2 + bx + c = 0 \qquad \text{has solutions} \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{220}$$

where if $b^2 - 4ac < 0$, the square root in question will be an imaginary number (i.e. a multiple of i). But it turns out that complex numbers allow us to find roots of arbitrary polynomials. To this end, the **fundamental theorem of algebra** states that:

$$\textbf{any polynomial of degree } n \textbf{ has exactly } n \textbf{ complex roots} \tag{221}$$

where the word "exactly" should be taken to mean "counted with multiplicities", as in Lecture 22.

Let's discuss one more feature of complex numbers, which is very useful in computations: the polar form. As we have seen, a complex number can be represented as a point in the plane via **Cartesian coordinates**:

$$z = a + ib \qquad \rightsquigarrow \qquad (a, b)$$

However, we can represent the same point via **polar coordinates**:

$$z = a + ib \qquad \rightsquigarrow \qquad (r, \theta)$$

where $r = \sqrt{a^2 + b^2}$ is the absolute value of z (i.e. the size of a circle around the origin where the point is located) and $\theta \in [0, 2\pi)$ is the angle between the point in question and the horizontal line:

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}} \qquad \Rightarrow \qquad \theta = \arccos\left(\frac{a}{\sqrt{a^2 + b^2}}\right)$$

The angle θ is called the **argument** of z, and is only defined up to adding arbitrary integer multiples of 2π (because θ and $\theta + 2\pi$ determine the same angle, there is an inherent ambiguity in the notion of "argument"). Since $(\cos\theta)^2 + (\sin\theta)^2 = 1$, we have $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$ and therefore:

$$\boxed{z = a + bi = r(\cos\theta + i\sin\theta)} \tag{222}$$

The above is called the **polar form** of z. Recall the Taylor series expansions of $\cos\theta$ and $\sin\theta$:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...$$
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...$$

which implies that:

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - ...$$

The right-hand side is the Taylor series of $e^{i\theta}$ (see (211)), so we obtain the following identity:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \tag{223}$$

The polar form (222) of a complex number can therefore be written in a nicer form as:

$$\boxed{z = a + bi = re^{i\theta}} \tag{224}$$

Since $|e^{i\theta}| = 1$, the polar form of a complex number can be said to separate its absolute value part (namely r) from its angular part (namely $e^{i\theta}$). One of the good things about the polar form is that, as opposed from the Cartesian form, it's handy for multiplying numbers and raising to powers:

$$z^n = r^n e^{in\theta} \tag{225}$$

and:

$$zz' = rr'e^{i(\theta+\theta')} \tag{226}$$

if $z = re^{i\theta}$ and $z' = r'e^{i\theta'}$. In other words, we have:

$$|zz'| = |z||z'|$$

$$\arg(zz') = \arg(z) + \arg(z')$$

(the sum of arguments formula only holds up to adding integer multiples of 2π). Formula (225) allows us to describe the roots of unity, i.e. solutions of:

$$z^n = 1 \tag{227}$$

If z is a real number, the only solution to this equation is $z = 1$ (and also $z = -1$ if n is even), but the equation in question has more solutions in the world of complex numbers. Specifically, as a consequence of the fundamental theorem of algebra (221), we expect the equation (227) to have n complex solutions. If $z = re^{i\theta}$, then combining formulas (225) and (227) gives us:

$$r^n e^{in\theta} = 1$$

Since r is a positive real number, we must have $r = 1$. Since θ is an angle, the equation above implies $n\theta$ to be an even integral multiple of 2π. Therefore, we conclude that the solutions of the equation (227), namely the n–**th roots of unity**, are:

$$1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, ..., e^{\frac{2(n-1)\pi i}{n}} \tag{228}$$

There's no need to go beyond the $(n-1)$–th multiple of $\frac{2\pi}{n}$, since then we will just recover the same numbers over again (this is simply because θ and $\theta + 2\pi$ represent the same angle).

Lecture 25 (*November 8*) *Reading: section 6.4–6.5*

Let us now consider eigenvalues and eigenvectors of symmetric matrices.

Fact 20. *A $n \times n$ symmetric matrix S has n real eigenvalues and n orthonormal eigenvectors.*

You can find a proof of the fact that the eigenvalues are real in the textbook, the bottom of page 339. For example, in the 2×2 case, a symmetric matrix:

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{229}$$

has characteristic polynomial:

$$p(\lambda) = \lambda^2 - \lambda(a+c) + ac - b^2$$

whose roots are:

$$\frac{a + c \pm \sqrt{(a+c)^2 - 4ac + 4b^2}}{2} = \frac{a + c \pm \sqrt{(a-c)^2 + 4b^2}}{2} \tag{230}$$

It is clear that these are real, because $(a-c)^2 + 4b^2 \geq 0$. As for the fact that the eigenvectors of a symmetric matrix S are orthogonal, suppose $\lambda \neq \lambda'$ are distinct eigenvalues with corresponding eigenvectors $\boldsymbol{v}, \boldsymbol{v}'$. Then:

$$\lambda \cdot \boldsymbol{v}^T \boldsymbol{v}' = (S\boldsymbol{v})^T \boldsymbol{v}' = \boldsymbol{v}^T S^T \boldsymbol{v}' = \boldsymbol{v}^T S \boldsymbol{v}' = \lambda' \cdot \boldsymbol{v}^T \boldsymbol{v}'$$

which implies $\boldsymbol{v}^T \boldsymbol{v}' = 0$. Once you know that the eigenvectors are orthogonal, they can be made orthonormal by rescaling them appropriately (since any multiple of an eigenvector is an eigenvector). As a consequence of the fact above, we may put the orthonormal eigenvectors $\boldsymbol{q}_1, ..., \boldsymbol{q}_n$ in a matrix:

$$Q = \left[\begin{array}{c|c|c} & & \\ \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \\ & & \end{array} \right]$$

which is, by definition, an orthogonal matrix. Then (177) implies that we may write:

$$S = Q \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} Q^{-1} = Q \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} Q^T \tag{231}$$

The last equality follows from the fact that $Q^{-1} = Q^T$, a general feature of orthogonal matrices.

Geometrically, equality (231) can be interpreted as the fact that the linear transformation corresponding to a symmetric matrix can always be factored as:

$$\Big(\text{rotation}\Big)\Big(\text{scaling}\Big)\Big(\text{inverse rotation}\Big) \tag{232}$$

Indeed, diagonal matrices correspond to scaling in the direction of the coordinate axes, while orthogonal matrices correspond to rotations in n–dimensional space (as an exercise, prove that any 2×2 orthogonal matrix must be literally equal to one of the rotation matrices on the top of page 47). Explicitly, formula (231) implies that we can always express a symmetric matrix as:

$$\boxed{S = \lambda_1 \boldsymbol{q}_1 \boldsymbol{q}_1^T + ... + \lambda_n \boldsymbol{q}_n \boldsymbol{q}_n^T} \tag{233}$$

(you can prove this by applying both sides of the identity to an arbitrary eigenvector $\boldsymbol{q}_i$, and then extending by linear combinations to arbitrary vectors).

Remark. *For an arbitrary (not necessarily symmetric) matrix A, the eigenvalues can be complex numbers. However, when this happens:*

The complex eigenvalues of any real matrix come in conjugate pairs (234)

For example, in a 2×2 example, the characteristic polynomial:

$$p(\lambda) = \lambda^2 - \lambda \cdot \mathrm{Tr} + \det$$

has roots:

$$\frac{\mathrm{Tr} \pm \sqrt{\mathrm{Tr}^2 - 4 \cdot \det}}{2}$$

If $\mathrm{Tr}^2 < 4 \cdot \det$, the roots are complex numbers, and they are manifestly conjugates of each other. Moreover, the eigenvectors corresponding to conjugate eigenvalues are also conjugate:

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \qquad \Rightarrow \qquad A\overline{\boldsymbol{v}} = \bar{\lambda}\bar{\boldsymbol{v}}$$

While in general, the eigenvalues and pivots of a square matrix are not directly related to each other (other than the fact that the eigenvalues and the pivots both have the same product, i.e. the determinant), for a symmetric matrix we have:

there are as many positive eigenvalues as positive pivots (235)

The analogous statement is true if "positive" were replaced by "negative".

Definition 21. *A symmetric matrix with all eigenvalues/pivots > 0 is called **positive definite**. A symmetric matrix with all eigenvalues/pivots ≥ 0 is called **positive semidefinite**.*

In general, if S is a positive definite (respectively semidefinite) matrix, then the "energy" [3]:

$$\boxed{\boldsymbol{v}^T S\boldsymbol{v} > 0} \qquad \text{(respectively } \boldsymbol{v}^T S\boldsymbol{v} \geq 0) \tag{236}$$

for any vector $\boldsymbol{v} \neq 0$. Indeed, just decompose $\boldsymbol{v}$ as a linear combination of the eigenvectors of S:

$$\boldsymbol{v} = c_1\boldsymbol{q}_1 + ... + c_n\boldsymbol{q}_n$$

and then (236) becomes (because the $\boldsymbol{q}_i$'s are orthonormal):

$$\boldsymbol{v}^T S\boldsymbol{v} = \sum_{1\leq i,j\leq n} c_i c_j \boldsymbol{q}_i^T S\boldsymbol{q}_j = \sum_{1\leq i,j\leq n} c_i c_j \lambda_j \boldsymbol{q}_i^T \boldsymbol{q}_j = \sum_{i=1}^{n} c_i^2 \lambda_i$$

which is positive (respectively non-negative) if $\lambda_1, ..., \lambda_n > 0$ (respectively $\lambda_1, ..., \lambda_n \geq 0$). Note: condition (236) is actually equivalent to the matrix S being positive definite (respectively semidefinitie). This gives a simple argument for why S and T being positive definite/semidefinite matrices implies that $\alpha S + \beta T$ is positive definite/semidefinite for any positive numbers α, β.

Remark. *We already know that a big source of symmetric matrices are $S = A^T A$ for an arbitrary matrix A. This matrix is positive semidefinite, because:*

$$\boldsymbol{v}^T S\boldsymbol{v} = \boldsymbol{v}^T A^T A\boldsymbol{v} = (A\boldsymbol{v})^T (A\boldsymbol{v}) = ||A\boldsymbol{v}||^2 \geq 0 \tag{237}$$

Moreover, $S = A^T A$ is positive definite precisely if $A\boldsymbol{v} \neq 0$ for all $\boldsymbol{v} \neq 0$, which happens if and only if A has linearly independent columns (i.e. have trivial nullspace). Conversely, any positive definite matrix is of the form $A^T A$ for some matrix A with linearly independent columns.

Let's work out in detail the case of a 2×2 symmetric matrix S, given by (229). Since the eigenvalues of S are given by (230), in order for them to both be positive, we would need $a + c > 0$ and:

$$a + c > \sqrt{(a-c)^2 + 4b^2} \quad \Rightarrow \quad (a+c)^2 > (a-c)^2 + 4b^2 \quad \Rightarrow \quad ac > b^2$$

So we conclude that:

$$\text{A } 2\times 2 \text{ symmetric matrix is positive definite precisely when Tr } S, \det S > 0 \tag{238}$$

As for the energy in the case when S is given by (229), we have:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 \tag{239}$$

It is a well-known fact in algebra that the expression above is positive for all x and y if $a > 0$ and $ac > b^2$, precisely as predicted by (238). In fact, you may recognize the equation "(239) = 1" as the equation of a conic in the plane: if S is positive definite, then this conic is an ellipse, while otherwise it's a hyperbola (the intermediary case, when one of the eigenvalues is 0, is a parabola).

Let's do all of this in a numerical example:

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

[3] The name comes from physics, in the case when $\boldsymbol{v}$ is the vector of momenta of a system of particles

Its characteristic polynomial is $\lambda^2-10\lambda+9$, whose roots are 9 and 1. The corresponding eigenvectors are $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, hence we conclude that:

$$S = QDQ^T$$

where:

$$D = \begin{bmatrix}9 & 0\\0 & 1\end{bmatrix} \qquad \text{and} \qquad Q = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

The ellipses corresponding to S and D are:

$$5x^2 + 8xy + 5y^2 = 1 \qquad \text{and} \qquad 9x^2 + y^2 = 1$$

The latter ellipse has its axes lined up with the coordinate axes. The former has its axes lined up with the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, which is consistent with the fact that going from D to S amounts to rotating the coordinate axes from $\boldsymbol{e}_1, \boldsymbol{e}_2$ to $\boldsymbol{v}_1, \boldsymbol{v}_2$, i.e. applying the linear transformation Q.

Lecture 26 (*November 13*) *Reading: section 7.1–7.2*

We will now study the **Singular Value Decomposition (SVD)** of a matrix, which is very useful in computations. Think of image processing: in general, this deals with a large rectangular table of pixels, and the color of every pixel is an integer. So the image is encoded by an $m \times n$ matrix A of integers, where m and n are two very large numbers which keep track of how many pixels we have vertically and horizontally. If the colors of all the pixels are random, then there's not much you can do to simplify the matrix A. But in real life, nearby pixels tend to have similar colors, and patterns emerge that allow you to conclude that the matrix A has a special form. An extreme example is where your image consists only of three colors separated by vertical lines, such as a flag:

$$A = \begin{bmatrix}1&1&2&2&2&2&2&3&3&3\\1&1&2&2&2&2&2&3&3&3\\1&1&2&2&2&2&2&3&3&3\\1&1&2&2&2&2&2&3&3&3\\1&1&2&2&2&2&2&3&3&3\\1&1&2&2&2&2&2&3&3&3\\1&1&2&2&2&2&2&3&3&3\\1&1&2&2&2&2&2&3&3&3\end{bmatrix}$$

Remembering the whole matrix is both memory and computational power-consuming, so you'd be more efficient by remembering instead the fact that A can be presented as the product:

$$A = \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1&1&2&2&2&2&2&3&3&3\end{bmatrix}$$

of a column vector and a row vector. In general, the goal of SVD is to write any rectangular matrix as a sum of such products (let r be the rank of A):

$$\boxed{A = \boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^T + ... + \boldsymbol{u}_r\sigma_r\boldsymbol{v}_r^T} \tag{240}$$

where each $\boldsymbol{u}_i$ is a column vector, each $\boldsymbol{v}_i^T$ is a row vector, and:

the positive numbers $\sigma_1, ..., \sigma_r > 0$ are called singular values

Let us now explain how to obtain the decomposition (240).

Fact 21. *The $m \times m$ matrix AA^T and the $n \times n$ matrix A^TA are both symmetric and positive semidefinite. These two matrices have the same set of positive eigenvalues; call them $\sigma_1^2, ..., \sigma_r^2$.*

You proved the latter statement in Problem Set 8 (also, it's not hard to prove that the number of positive eigenvalues is equal to $r = \text{rank } A$). Since symmetric matrices have orthonormal eigenvectors, we can choose:

$$\begin{aligned} &\textbf{left singular vectors} \quad \boldsymbol{u}_1, ..., \boldsymbol{u}_r, ..., \boldsymbol{u}_m \in \mathbb{R}^m \\ &\textbf{right singular vectors } \boldsymbol{v}_1, ..., \boldsymbol{v}_r, ..., \boldsymbol{v}_n \in \mathbb{R}^n \end{aligned}$$

such that the $\boldsymbol{u}_i$'s are orthonormal and the $\boldsymbol{v}_i$'s are orthonormal, and:

$$AA^T\boldsymbol{u}_i = \sigma_i^2\boldsymbol{u}_i \qquad \text{and} \qquad A^TA\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i \tag{241}$$

(the formulas above even hold for $i > r$, if we let $\sigma_{r+1}, \sigma_{r+2}, ... = 0$). But we actually want the left and right singular vectors to also be connected, via the following formulas:

$$\boxed{A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i} \qquad \text{and} \qquad \boxed{A^T\boldsymbol{u}_i = \sigma_i\boldsymbol{v}_i} \tag{242}$$

which also hold for all i, even $i > r$. The numbers $\sigma_1, ..., \sigma_r$ are called **singular values**. It is easy to see that (242) implies both (241) and (240): the former implication is an easy exercise, while the latter follows by multiplying both sides of (240) with any given vector $\boldsymbol{v}_i$ and using that $\boldsymbol{v}_j^T\boldsymbol{v}_i = 0$ if $j \neq i$. Note that the singular value decomposition (240) can be written in compact form as:

$$\boxed{A = U\Sigma V^T} \tag{243}$$

where:

$$U = \left[\begin{array}{c|c|c} \boldsymbol{u}_1 & ... & \boldsymbol{u}_m \end{array}\right] \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & \dots \\ 0 & \ddots & 0 & 0 & \dots \\ 0 & 0 & \sigma_r & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad V = \left[\begin{array}{c|c|c} \boldsymbol{v}_1 & ... & \boldsymbol{v}_n \end{array}\right]$$

Remark. *From* (242)*, it follows that $\boldsymbol{u}_1$,...,$\boldsymbol{u}_r$ form a basis of the column space of A, and $\boldsymbol{v}_1$,...,$\boldsymbol{v}_r$ form a basis of the row space. The orthogonality of the four subspaces therefore implies that $\boldsymbol{u}_{r+1}, ..., \boldsymbol{u}_m$ form a basis of the left nullspace of A, while $\boldsymbol{v}_{r+1}, ..., \boldsymbol{v}_n$ form a basis of the nullspace.*

Since V is an orthogonal matrix, we have $V^T = V^{-1}$, so (243) implies:

$$\boxed{AV = U\Sigma} \tag{244}$$

which is just another way to write the first identity in (242). As for the second identity in (242), we can obtain it as follows. First transpose relation (243):

$$A^T = V\Sigma^T U^T \tag{245}$$

Since U is an orthogonal matrix, we have $U^T = U^{-1}$, hence:

$$A^T U = V\Sigma^T \tag{246}$$

Therefore, the SVD of the transpose of A is obtained by transposing the SVD of A.

Let's do an example of SVD, which will also show us that the procedure works, for the matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

We first need to compute the singular values, which will be the square roots of the eigenvalues of:

$$A^T A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

The characteristic polynomial of this matrix is:

$$p(\lambda) = \det \begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} = (17-\lambda)^2 - 8^2$$

and its roots will be precisely those numbers for which $17 - \lambda = \pm 8$, so:

$$\begin{aligned} 17 - \sigma_1^2 = -8 \quad &\Rightarrow \quad \sigma_1^2 = 25 \quad \Rightarrow \quad \sigma_1 = 5 \\ 17 - \sigma_2^2 = 8 \quad &\Rightarrow \quad \sigma_2^2 = 9 \quad \Rightarrow \quad \sigma_2 = 3 \end{aligned}$$

(it is customary to order your signular values $\sigma_1, ..., \sigma_r$ from the greatest to the smallest). Next, you need to find the right singular vectors, which will be eigenvectors of the matrix $A^T A$:

$$\begin{aligned} \boldsymbol{v}_1 \in N\left(\begin{bmatrix} 17-\sigma_1^2 & 8 \\ 8 & 17-\sigma_1^2 \end{bmatrix}\right) = N\left(\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix}\right) \quad &\Rightarrow \quad \boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \boldsymbol{v}_2 \in N\left(\begin{bmatrix} 17-\sigma_2^2 & 8 \\ 8 & 17-\sigma_2^2 \end{bmatrix}\right) = N\left(\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}\right) \quad &\Rightarrow \quad \boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

Note that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthonormal (indeed, eigenvectors of symmetric matrices are always orthogonal, and we have scaled them so as to have length 1). Now let's compute the left singular vectors. Formula (242) forces us to take

$$\begin{aligned} \boldsymbol{u}_1 &= \frac{1}{5} A\boldsymbol{v}_1 = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \boldsymbol{u}_2 &= \frac{1}{3} A\boldsymbol{v}_2 = \frac{1}{3} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \end{aligned}$$

The vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are also orthonormal (this follows from the fact that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are) but we need to complete them to an orthonormal basis of $\mathbb{R}^3$. This is achieved by Gram-Schmidt. Start with any vector, say:

$$\boldsymbol{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and construct the vector:

$$\boldsymbol{w}_3 = \boldsymbol{a}_3 - \text{proj}_{\boldsymbol{u}_1}\boldsymbol{a}_3 - \text{proj}_{\boldsymbol{u}_2}\boldsymbol{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot 0 - \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \cdot \frac{4}{3\sqrt{2}} = \frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

The vector $\boldsymbol{w}_3$ thus constructed is orthogonal to both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, so we need to renormalize it:

$$\boldsymbol{u}_3 = \frac{\boldsymbol{w}_3}{||\boldsymbol{w}_3||} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

With this in mind, we conclude the following SVD for the matrix A:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

So in general, what can we say about a rank 1 matrix? The answer is that it will be of the form:

$$A = \boldsymbol{u}\sigma\boldsymbol{v}^T \tag{247}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are length 1 vectors, and σ is a number. In other words, any column of a rank 1 matrix is a multiple of $\boldsymbol{u}$ and any row is a multiple of $\boldsymbol{v}$. With this in mind, the singular value decomposition (240) can be thought of as:

SVD writes any matrix as a sum of rank 1 matrices (248)

where the columns (respectively rows) of the rank 1 matrices involved are pairwise orthogonal. This is good to keep in mind, since it can save you a lot of time when doing SVD on a matrix where it's intuitively pretty obvious how to break it up as a sum of rank 1 matrices. For example:

$$A = \begin{bmatrix} 0 & 5 & 0 \\ -3 & 0 & 0 \end{bmatrix}$$

If you started off by doing the usual algorithm (compute the eigenvalues and eigenvectors of the 3×3 matrix A^TA) you might find yourself doing a lot of work. You can save a lot of time by observing that A decomposes as:

$$A = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$$

Each of the two matrices above has rank 1, because all of their columns are multiples of a single vector. Even more explicitly, we have:

$$A = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\boldsymbol{u}_1} \cdot \underbrace{5}_{\sigma_1} \cdot \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{\boldsymbol{v}_1} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\boldsymbol{u}_2} \cdot \underbrace{3}_{\sigma_2} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\boldsymbol{v}_2}$$

However, when using this shortcut, you need to make sure that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, ...$ you construct are orthonormal (and also that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ...$ are orthonormal).

Lecture 27 (*November 15*) *Reading: section 7.4*

In practice, the singular value decomposition (240) of a matrix gives us a computationally manageable way to compute how A acts on arbitrary vectors:

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n \tag{249}$$

(since the $\boldsymbol{v}_i$ are orthogonal, recall that $c_i = \boldsymbol{v}_i^T\boldsymbol{v}$). Then we have:

$$A\boldsymbol{v} = \sigma_1 c_1\boldsymbol{u}_1 + ... + \sigma_r c_r\boldsymbol{u}_r \tag{250}$$

It is customary to order your singular values $\sigma_1 \geq ... \geq \sigma_r > 0$. In this case, you can compare (249) with (250) to get:

$$\frac{||A\boldsymbol{v}||}{||\boldsymbol{v}||} = \sqrt{\frac{\sigma_1^2c_1^2 + ... + \sigma_r^2c_r^2}{c_1^2 + ... + c_r^2 + ... + c_n^2}} \leq \sigma_1$$

Moreover, we get equality in the inequality above precisely when $\boldsymbol{v}$ is a multiple of $\boldsymbol{v}_1$. Therefore:

$$\sigma_1 \textbf{ is the maximum of } \frac{||A\boldsymbol{v}||}{||\boldsymbol{v}||}, \textbf{ achieved for } \boldsymbol{v} = \boldsymbol{v}_1 \tag{251}$$

which is an intrinsic characterization of the largest singular value. Specifically, we are saying that the largest singular value is equal to the **norm** of the matrix:

$$||A|| := \max_{\boldsymbol{w}\in\mathbb{R}^n\setminus 0} \frac{||A\boldsymbol{w}||}{||\boldsymbol{w}||} \in \mathbb{R}_+ \tag{252}$$

The norm of a matrix satisfies the usual "triangle inequality" satisfied by the norms of vectors:

$$||A + B|| \leq ||A|| + ||B|| \tag{253}$$

(proof: for any vector $\boldsymbol{w}$, we have $||(A+B)\boldsymbol{w}|| \leq ||A\boldsymbol{w}|| + ||B\boldsymbol{w}|| \leq ||A||\cdot||\boldsymbol{w}|| + ||B||\cdot||\boldsymbol{w}|| = (||A|| + ||B||)\cdot||\boldsymbol{w}||$, which implies (253)) as well as a similar inequality when multiplying matrices:

$$||AB|| \leq ||A||\cdot||B|| \tag{254}$$

(proof: for any vector $\boldsymbol{w}$, we have $||AB\boldsymbol{w}|| \leq ||A||\cdot||B\boldsymbol{w}|| \leq ||A||\cdot||B||\cdot||\boldsymbol{w}||$, which implies (254)). You can obtain norm-like descriptions of the other singular values of a matrix, as follows:

$$\sigma_2 \textbf{ is the maximum of } \frac{||A\boldsymbol{v}||}{||\boldsymbol{v}||} \textbf{ among those } \boldsymbol{v} \perp \boldsymbol{v}_1, \textbf{ achieved for } \boldsymbol{v} = \boldsymbol{v}_2 \tag{255}$$

$$\sigma_3 \textbf{ is the maximum of } \frac{||A\boldsymbol{v}||}{||\boldsymbol{v}||} \textbf{ among those } \boldsymbol{v} \perp \boldsymbol{v}_1, \boldsymbol{v}_2, \textbf{ achieved for } \boldsymbol{v} = \boldsymbol{v}_3 \tag{256}$$

etc. In fact, this recursive procedure defines the singular values σ_i and the right singular vectors $\boldsymbol{v}_i$ (from which the left singular vectors can be deduced from (242)): first σ_1 and $\boldsymbol{v}_1$ are defined by (251), then σ_2 and $\boldsymbol{v}_2$ are defined by (255), then σ_3 and $\boldsymbol{v}_3$ are defined by (256) etc.

Geometrically, the SVD in the form (243) means that any matrix A can be factored as:

$$\Big(\text{rotation}\Big)\Big(\text{scaling}\Big)\Big(\text{another rotation}\Big) \tag{257}$$

Compare this with (232) for symmetric matrices: the novelty in (257) is that the two rotations, which correspond to the orthogonal matrices U and V^T respectively, are now completely different linear transformations acting on two different vector spaces. This is natural, since A corresponds to a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Meanwhile, the two rotations in (232) are inverses to each other, which is natural because the symmetric matrix therein had the same domain and target.

You can find a picture of the factorization (257) on page 392 of the textbook: the two rotations do not change any given shape, they just rotate it around space. But the scaling in the middle does change shapes by dilations across various axes: circles become ellipses, squares become rectangles etc. In the case of 2×2 matrices, the factorization explicitly says that:

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}}_{V^T}$$

Now let us recall formula (245), which explains that the transpose of a matrix A has the transposed singular value decomposition. So what can we say about the "inverse" of a matrix? The reason why I put the word "inverse" in quotes is that we are in the rectangular case, where a proper inverse most likely does not exist. However, if it did, then from formula (242) we would expect:

$$\text{“}A^{-1}\text{”}(\boldsymbol{u}_i) = \frac{\boldsymbol{v}_i}{\sigma_i} \tag{258}$$

The goal of the following is to define a notion of "inverse" which satisfies the equation above.

Definition 22. *The **pseudo-inverse** of an $m \times n$ matrix $A = U\Sigma V^T$ is the $n \times m$ matrix:*

$$A^+ = V\Sigma^+ U^T \tag{259}$$

where $\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 & 0 & \dots \\ 0 & \ddots & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma_r} & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

The pseudo-inverse coincides with the inverse in the cases when the latter exists, i.e. $m = n = r$. In general, the pseudo-inverse satisfies the following inverse-like properties:

$$A^+A = V\Sigma^+U^TU\Sigma V^T = V(\Sigma^+\Sigma)V^T = V \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] V^T$$

$$AA^+ = U\Sigma V^TV\Sigma^+U^T = U(\Sigma\Sigma^+)U^T = U \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] U^T$$

If the block matrix in the right-hand sides were just equal to the identity matrix I, then A and A^+ would be bona fide inverses. However, as such, the block matrix is simply the projection matrix onto the first r coordinate vectors. And when we conjugate this block matrix by the orthogonal matrices V and U, respectively, we conclude that:

$$A^+A = \text{projection onto } C(A^T) \tag{260}$$

$$AA^+ = \text{projection onto } C(A) \tag{261}$$

This happens because the row space (respectively the column space) of the matrix A is spanned by the first r vectors of the matrix V (respectively U), as in the Remark in the previous Lecture.

As an example, let's compute the pseudo-inverse of the matrix:

$$A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

This matrix has rank 1, so its SVD is computed just like at the end of the previous lecture:

$$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \cdot 25 \cdot \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} \quad \Rightarrow \quad A^+ = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \cdot \frac{1}{25} \cdot \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

Pseudo-inverses have a nice application to least squares: remember that the closest approximation to the system $A\boldsymbol{v} = \boldsymbol{b}$ is $\boldsymbol{v} = (A^TA)^{-1}A^T\boldsymbol{b}$, but we always made the caveat that the columns of A need to be independent in order for A^TA to be invertible. If A has dependent columns, while we cannot write $(A^TA)^{-1}$, we can use the pseudo-inverse to give us the following solution:

$$\textbf{the closest } A\boldsymbol{v} \textbf{ is to } \boldsymbol{b} \textbf{ is achieved for } \boxed{\boldsymbol{v}^+ = A^+\boldsymbol{b}} \tag{262}$$

Such a $\boldsymbol{v}$ is not unique, basically because the fact that A has dependent columns implies that its nullspace is non-trivial (so to any particular solution $\boldsymbol{v}$, you could add any vector in the nullspace of A, and you would still get a solution). But to see that the particular solution $\boldsymbol{v}^+$ in (262) does indeed satisfy $A\boldsymbol{v}^+ = \text{proj}_{C(A)}\boldsymbol{b}$, just apply A to both sides of the equation in the box. You get:

$$A\boldsymbol{v}^+ = (AA^+)\boldsymbol{b} = \text{proj}_{C(A)}\boldsymbol{b}$$

as a consequence of (261). So $A\boldsymbol{v}^+$ is precisely the projection of $\boldsymbol{b}$ onto the column space of A, as required by the geometry of least squares approximation.

Let's now consider one last application of SVD's in the case of square matrices (i.e. $m = n$):

Definition 23. *The **polar decomposition** of an $n \times n$ matrix A is:*

$$A = QS \tag{263}$$

where Q is orthogonal, and S is positive semidefinite.

The name stems from the fact that in the 1×1 case, any number a has a polar form (224): the number $e^{i\theta}$ has absolute value 1 so it behaves like an orthogonal 1×1 matrix, and the number $r \geq 0$ is a positive semidefinite 1×1 matrix. In general, you can obtain the polar decomposition of a matrix from its singular value decomposition, as follows:

$$A = U\Sigma V^T = \underbrace{(UV^T)}_{Q} \cdot \underbrace{(V\Sigma V^T)}_{S} \tag{264}$$

Indeed, the product of orthogonal matrices is orthogonal (it is key here that $m = n$, i.e. this all applies to square matrices), which means that $Q := UV^T$ is orthogonal. Moreover, $S := V\Sigma V^T$ is the usual formula for the diagonalization of a symmetric matrix, as we have seen in (231). The fact that the eigenvalues of S, i.e. the diagonal entries of Σ, are non-negative is precisely equivalent with saying that S is positive semidefinite. The eigenvalues of S are the singular values of A.

Lecture 28 (*November 18*) *Reading: section 12.1*

We will now start discussing major applications of linear algebra: probability and statistics. First of these is **probability**, which studies the likelihood of future events. Suppose we're in a situation with n possible outcomes $x_1, ..., x_n$, which arise with probabilities $p_1, ..., p_n$, respectively. Then:

$$p_1, ..., p_n > 0 \qquad \text{and} \qquad p_1 + ... + p_n = 1 \tag{265}$$

Definition 24. *The **mean** is the sum of the possible outcomes, weighted by their probabilities:*

$$\mu = p_1 x_1 + ... + p_n x_n \tag{266}$$

*and the **variance** is:*

$$\Sigma = p_1(x_1 - \mu)^2 + ... + p_n(x_n - \mu)^2 \tag{267}$$

*The square root of the variance, namely $\sigma = \sqrt{\Sigma}$, is called the **standard deviation**.*

The mean is sometimes called the **expected value**, which is the plain English significance of this notion. The variance is a positive quantity which measures how far away from the mean our outcomes are. Clearly, the variance is 0 only if there exists a single outcome with probability 1.

The situation described above is where there are finitely many (or discrete) outcomes. But we could also study a quantity which takes infinitely many (or continuous) values, for example temperature. In this case, the probability is encoded not in a finite set of numbers, but in a function:

$$p(x) : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$$

called **probability distribution**. In this context, you shouldn't measure the probability of your quantity being exactly equal to x (for example, it makes little sense to measure the probability that temperature will be exactly 62.746 degrees) but you measure instead the probability that the quantity lies in a certain interval $[a, b]$ (for example, you measure the probability that temperature falls between 62 and 63 degrees):

$$\text{Prob}(x \in [a, b]) = \int_a^b p(x)dx \tag{268}$$

The reason for the integral is that it is the continuous analogue of a sum: if you subdivide the interval $[a, b]$ into many (say N very large) small intervals, then the integral in (268) is infinitesimally close to the sum of the probabilities that x lies in the N small intervals. In this continuous context, condition (265) is replaced by:

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{269}$$

while the mean and variance (266) and (267) must be defined by integrals:

$$\mu = \int_{-\infty}^{\infty} p(x)x dx \tag{270}$$

$$\Sigma = \int_{-\infty}^{\infty} p(x)(x - \mu)^2 dx \tag{271}$$

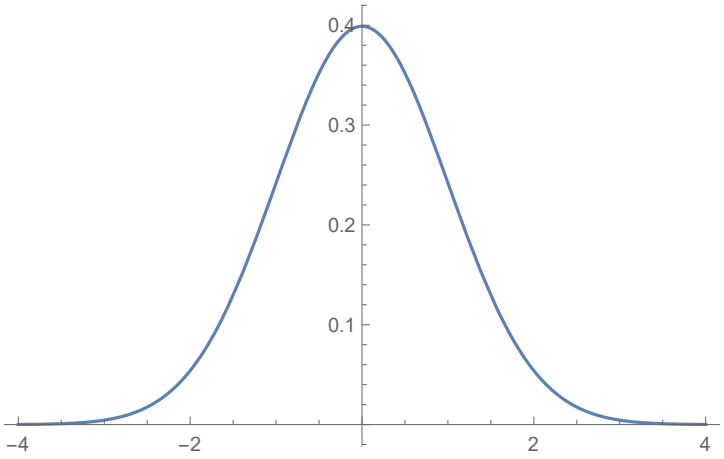
The most basic example of probability distribution is the uniform one, where the variable can lie in an interval (say $[0, c]$) with constant probability:

$$p(x) = \begin{cases} \frac{1}{c} & \text{if } x \in [0; c] \\ 0 & \text{otherwise} \end{cases}$$

However, this probability distribution does not really show up in nature. Real-life probabilities do not like the sharp cut-offs that the function above experiences at the endpoints of the interval $[0, c]$. Perhaps even more importantly, real-life probability distributions tend to be higher around the mean and lower away from the mean. The standard example of such a behavior is the **normal (or Gaussian) distribution**:

$$p(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \tag{272}$$

whose graph (sometimes called a "bell curve") is plotted below. As always, the probability that the variable x lies in a given interval $[a, b]$ is equal to the integral of $p(x)$ (or the area under the graph) from $x = a$ to $x = b$.



The reason for dividing by the factor $\sqrt{2\pi}$ is because you want the total probability to be equal to 1, i.e. (269). The mean and variance of the normal distribution (272) are given by:

$$\mu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx = 0$$

$$\Sigma = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx = 1$$

More generally, the normal probability distribution with arbitrary mean $\mu \in \mathbb{R}$ and arbitrary variance $\Sigma > 0$ is given by the function:

$$p(x) = \frac{e^{-\frac{(x-\mu)^2}{2\Sigma}}}{\sqrt{2\pi\Sigma}} \tag{273}$$

Normal distributions are extremely important, because of the central limit theorem: if you perform an experiment $N \to \infty$ times, the mean of your samples tends to look as if it came from a normal

distribution. For example, if you flip a coin N times, the probability of getting exactly k heads is close to the probability that $x \in [0, k]$ for the normal distribution (273) with mean $\mu = \frac{N}{2}$ and variance $\Sigma = \frac{N}{4}$. In the error analysis of the probability distribution (273), the probability that:

$$\begin{aligned}
&x \text{ is at most } \ \sigma \text{ away from the mean is } \sim 68\% \\
&x \text{ is at most } 2\sigma \text{ away from the mean is } \sim 95\% \\
&x \text{ is at most } 3\sigma \text{ away from the mean is } \sim 99.7\% \\
&x \text{ is at most } 4\sigma \text{ away from the mean is } \sim 99.99\% \\
&x \text{ is at most } 5\sigma \text{ away from the mean is } \sim 99.9999\%
\end{aligned}$$

You can actually see this in the graph of the bell curve above: about 99.7% of the area under the graph of the bell curve is between $x = -3$ and $x = 3$ etc. Let's summarize the discussion above.

Definition 25. *A **random variable** is a quantity X that takes values in $\mathbb{R}$, which is either:*

- *discrete, so takes finitely many values $x_1, ..., x_n$ with probabilities $p_1, ..., p_n$, or*
- *continuous, associated to a probability distribution $p(x)$.*

In either case, the mean (or expected value) is denoted by:

$$E[X] = \mu \tag{274}$$

and is defined by either (266) or (270). Then the variance is:

$$E\Big[(X - \mu)^2\Big] = \Sigma \tag{275}$$

and is given by either (267) or (271).

Lecture 29 (*November 20*) *Reading: section 12.2*

The true power of linear algebra in probability comes about when we run more than one experiment at once, and we want to compare the outcomes. The notion that measures this is the following.

Definition 26. *Given two random variables X and Y, their **covariance** is the quantity:*

$$\Sigma_{XY} = E\Big[(X - E[X])(Y - E[Y])\Big] \tag{276}$$

*Note that Σ_{XX} is just the **variance** of the random variable X itself.*

Let's start with the case of discrete probabilities, so X and Y both take a finite number of values with certain probabilities. The whole point of covariance is to measure the extend to which X and

Y are correlated with each other. So it does not suffice to simply know the probabilities of the various values of X and Y separately, but instead you need to know the **joint probabilities**:

$$p_{ij} \text{ is the probability that } (X, Y) = (x_i, y_j) \tag{277}$$

where $x_1, x_2, ...$ are the possible values of X and $y_1, y_2, ...$ are the possible values of Y.

Example 4. *Let's assume that X and Y are coin flips, and their possible values are heads (H) and tails (T). If the two coin flips are independent, then the joint probabilities are:*

$$p_{HH} = p_{TT} = p_{HT} = p_{TH} = \frac{1}{4}$$

But if the random variables are two coin flips which are forced to always land on the same side (think something involving magnets), then the joint probabilities are:

$$p_{HH} = p_{TT} = \frac{1}{2} \quad \textit{and} \quad p_{HT} = p_{TH} = 0$$

The means of the two random variables X and Y are:

$$\mu = \sum_{i,j} p_{ij} x_i$$

$$\nu = \sum_{i,j} p_{ij} y_j$$

and therefore the covariance of the two random variables is:

$$\Sigma_{XY} = \sum_{i,j} p_{ij} (x_i - \mu)(y_j - \nu) \tag{278}$$

If X and Y are continuous random variables, their **joint probability distribution** is a function:

$$p(x, y) : \mathbb{R}^2 \to \mathbb{R}_{\geq 0} \tag{279}$$

which encodes probability in the sense that:

$$\text{Prob}(X \in [a, b] \text{ and } Y \in [c, d]) = \int_a^b \int_c^d p(x, y) dy dx \tag{280}$$

With this in mind, the covariance of the random variables X and Y is:

$$\Sigma_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y)(x - \mu)(y - \nu) dx dy \tag{281}$$

where μ and ν are the means of X and Y, defined by (270).

Remark. *The notion* (279) *is necessary because the random variables X and Y may be dependent on each other: mathematically, this means that where X is on the real line might impact where Y is. For example, if $X = Y$, then X and Y cannot be at different points of the real line, so in this case the function $p(x, y)$ is concentrated on the diagonal in $\mathbb{R}^2$.*

An important special case is when two variables X and Y are independent of each other. In the discrete case, this means that the probabilities (277) are given by $p_{ij} = a_i b_j$, where $a_1, a_2, ...$ and $b_1, b_2, ...$ are the probabilities of X and Y separately. In the continuous case, the fact that X and Y are independent means that the joint probability distribution (279) is given by:

$$p(x, y) = a(x)b(y)$$

where a and b are the probability distributions of X and Y, respectively. In this case, the position of X on the real line has no impact on the position of Y and vice versa. The covariance (280) is:

$$\Sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x)b(y)(x-\mu)(y-\nu)dxdy = \left(\int_{-\infty}^{\infty} a(x)(x-\mu)dx\right)\left(\int_{-\infty}^{\infty} b(y)(y-\nu)dy\right) = 0$$

(the last equality is a consequence of (269) and (270)). We conclude that the covariance of two independent random variables is 0, which is just the intermediate case of the following principle:

- **the more positive the covariance, the more correlated the random variables are**
- **the more negative the covariance, the more anti–correlated the random variables**

For any random variables X and Y (discrete or continuous), the Cauchy-Schwartz inequality states:

$$|\Sigma_{XY}| \leq \sqrt{\Sigma_{XX}\Sigma_{YY}} \tag{282}$$

In other words, the covariance is bounded by the geometric mean of the variances. Therefore, if the variances of X and Y are fixed, the maximum covariance happens when $X = Y$ (the two random variables are as correlated as possible) and the minimum covariance happens when $X = -Y$ (the two random variables are as anti–correlated as possible).

Definition 27. *The covariance matrix of two random variables X and Y is:*

$$K = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix} \tag{283}$$

Since $\Sigma_{XY} = \Sigma_{YX}$, the covariance matrix is symmetric. But it is actually positive semidefinite, because its trace is positive (as Σ_{XX} and Σ_{YY} are the variances of X and Y) and its determinant is ≥ 0 according to (282). Moreover, the covariance matrix is positive definite unless its determinant is 0, which happens if the variables X and Y are **perfectly correlated** (e.g. when $X = Y$).

Let's compute the covariance matrix of discrete random variables in terms of the probabilities (277):

$$K = \sum_{i,j=1}^{k} p_{ij} \begin{bmatrix} (x_i-\mu)^2 & (x_i-\mu)(y_j-\nu) \\ (x_i-\mu)(y_j-\nu) & (y_j-\nu)^2 \end{bmatrix} = \sum_{i,j=1}^{k} p_{ij} \underbrace{\begin{bmatrix} x_i-\mu \\ y_j-\nu \end{bmatrix}\begin{bmatrix} x_i-\mu & y_j-\nu \end{bmatrix}} \tag{284}$$

The underbraced term is a positive semidefinite symmetric matrix, since it is of the form $\boldsymbol{v}\boldsymbol{v}^T$. The p_{ij}'s are non-negative numbers, and it is easy to show that a non-negative linear combination of positive semidefinite matrices is positive semidefinite (for example, using the energy criterion (236)). This gives another proof of the fact that the covariance matrix is positive semidefinite.

Definition 28. *Given n random variables $X_1, ..., X_n$, their covariance matrix is:*

$$K = \begin{bmatrix} \Sigma_{X_1X_1} & \dots & \Sigma_{X_1X_n} \\ \vdots & \ddots & \vdots \\ \Sigma_{X_nX_1} & \dots & \Sigma_{X_nX_n} \end{bmatrix} \tag{285}$$

The covariance matrix is a useful tool for determining how (in)dependent a collection of random variables is, for example if $X_1 =$ age, $X_2 =$ height, $X_3 =$ weight among a population. In fact, it is often convenient to put all of the random variables in a vector, called a **random vector**:

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \tag{286}$$

Let $p_1, p_2, ...$ to be the probability that $\boldsymbol{X}$ takes a certain number of vector values $\boldsymbol{x}_1, \boldsymbol{x}_2, ...$, where $\boldsymbol{x}_i = \begin{bmatrix} x_{1,i} \\ \vdots \\ x_{n,i} \end{bmatrix}$. In the setting of two random variables in (277), the probabilities in question were denoted by $p_{ij} = \text{Prob}\left(\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x_i \\ y_j \end{bmatrix}\right)$. Then the mean of the random vector (286) is:

$$\boldsymbol{\mu} = \sum_i p_i \boldsymbol{x}_i$$

and the covariance matrix is:

$$K = \sum_i p_i \begin{bmatrix} (x_{1,i} - \mu_1)^2 & \dots & (x_{1,i} - \mu_1)(x_{n,i} - \mu_n) \\ \vdots & \ddots & \vdots \\ (x_{1,i} - \mu_1)(x_{n,i} - \mu_n) & \dots & (x_{n,i} - \mu_n)^2 \end{bmatrix} =$$

$$= \sum_i p_i \begin{bmatrix} x_{1,i} - \mu_1 \\ \vdots \\ x_{n,i} - \mu_n \end{bmatrix} \begin{bmatrix} x_{1,i} - \mu_1 \\ \vdots \\ x_{n,i} - \mu_n \end{bmatrix}^T = \sum_i p_i (\boldsymbol{x}_i - \boldsymbol{\mu})(\boldsymbol{x}_i - \boldsymbol{\mu})^T = E\Big[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T\Big] \tag{287}$$

Apart from the fact that vector and matrix notation gives the neat formula (287) for the covariance matrix, the equality above proves that the covariance matrix of any number of random variables is positive semidefinite. Moreover, any linear combination of the random variables $X_1, ..., X_n$:

$$X = \boldsymbol{c}^T \boldsymbol{X} \qquad \text{for some} \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is a random variable (for example, in the age–height–weight example, X could be $2 \cdot \text{age} - 3 \cdot \text{height} + 7 \cdot \text{weight}$). The variance of X can be recovered from the covariance matrix K as:

$$E\Big[(X - \mu)^2\Big] = E\Big[(X - \mu)(X - \mu)^T\Big] = E\Big[(\boldsymbol{c}^T\boldsymbol{X} - \boldsymbol{c}^T\boldsymbol{\mu})(\boldsymbol{c}^T\boldsymbol{X} - \boldsymbol{c}^T\boldsymbol{\mu})^T\Big] =$$
$$= E\Big[(\boldsymbol{c}^T\boldsymbol{X} - \boldsymbol{c}^T\boldsymbol{\mu})(\boldsymbol{X}^T\boldsymbol{c} - \boldsymbol{\mu}^T\boldsymbol{c})\Big] = E\Big[\boldsymbol{c}^T(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X}^T - \boldsymbol{\mu}^T)\boldsymbol{c}\Big] = \boldsymbol{c}^T K \boldsymbol{c} \tag{288}$$

which is precisely the energy of the vector $\boldsymbol{c}$ with respect to the positive semidefinite covariance matrix K. The matrix K is positive definite unless the energy is 0 for a non-zero vector $\boldsymbol{c}$. Hence:

the covariance matrix is positive definite, unless

the random variables $X_1, ..., X_n$ are dependent

where "dependent" means that there exists some linear combination of $X_1, ..., X_n$ which is constant (since the constant random variables are precisely those with 0 variance).

Example 5. *Here's a quick application of this matrix formalism. Suppose you have random variables X and Y, with means μ and ν, and variances Σ_{XX} and Σ_{YY}. What are the mean and variance of the random variable $Z = X + Y$? You can compute these by writing:*

$$Z = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

The mean of Z is just:

$$\rho = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} = \mu + \nu$$

which is perhaps not so surprising. However, the variance of Z is given by formula (288)*:*

$$\Sigma_{ZZ} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{XY} & \Sigma_{YY} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Sigma_{XX} + \Sigma_{YY} + 2\Sigma_{XY}$$

Thus, the variance of a sum of random variables involves not only the sum of the individual variances, but also a cross-term involving the covariance of the two variables.

In this language, we can ask what is the meaning of diagonalizing the covariance matrix:

$$K = QDQ^T \tag{289}$$

where D is a diagonal matrix with non-negative eigenvalues, and Q has orthonormal columns. This procedure is called **principal component analysis (PCA)** in probability and statistics. Mathematically, this entails defining the following random vector:

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \boldsymbol{Y} := Q^T \boldsymbol{X} \qquad \text{whose mean will be} \qquad \boldsymbol{\nu} = Q^T \boldsymbol{\mu} \tag{290}$$

This simply means that the entries $Y_1, ..., Y_n$ of $\boldsymbol{Y}$ are all random variables, given by certain linear combinations of the random variables that make up $\boldsymbol{X}$. In this case, the covariance matrix corresponding to the random vector $\boldsymbol{Y}$ is:

$$E\Big[(\boldsymbol{Y} - \boldsymbol{\nu})(\boldsymbol{Y} - \boldsymbol{\nu})^T\Big] = E\Big[Q^T(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T Q\Big] = Q^T K Q = D = \begin{bmatrix} \Sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Sigma_n \end{bmatrix}$$

Since D is a diagonal matrix, this implies that the random variables $Y_1, ..., Y_n$ are mutually independent (since their pairwise covariances are 0), and that Σ_i is the variance of Y_i. Therefore, the diagonalization (289) corresponds to finding n linear combinations of the random variables $X_1, ..., X_n$, denoted by $Y_1, ..., Y_n$ above, which are pairwise independent random variables.

Lecture 30 (*November 22*) *Review of the previous two lectures*

Lecture 31 (*November 25*) *Reading: section 12.3*

Last time, we discussed random vectors and covariance matrices in the discrete case, but you probably won't be surprised to hear that the formalism can be extrapolated to continuous probability distributions. Generalizing (279), a function:

$$p(x_1, ..., x_n) : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \tag{291}$$

is called the joint probability distribution of a random vector $\boldsymbol{X}$ such as (286). The covariance matrix of this random vector is given by (287), if one replaces the sum by an integral:

$$K = E\Big[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T\Big] = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} p(x_1, ..., x_n) \begin{bmatrix} x_1 - \mu_1 \\ \vdots \\ x_n - \mu_n \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ \vdots \\ x_n - \mu_n \end{bmatrix}^T dx_1...dx_n \tag{292}$$

where:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} p(x_1, ..., x_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} dx_1...dx_n \tag{293}$$

A rather simple case is a random vector $\boldsymbol{Y}$ consisting of n independent normal distributions $Y_1, ..., Y_n$, i.e. consider the following joint probability distribution:

$$p(y_1, ..., y_n) = \frac{e^{-\frac{(y_1-\nu_1)^2}{2\Sigma_1} - ... - \frac{(y_n-\nu_n)^2}{2\Sigma_n}}}{\sqrt{(2\pi)^n \Sigma_1 ... \Sigma_n}} \tag{294}$$

(we denote them by y's instead of x's to avoid confusions later on). As a consequence of the identity:

$$-\frac{(y_1 - \nu_1)^2}{2\Sigma_1} - ... - \frac{(y_n - \nu_n)^2}{2\Sigma_n} = -\frac{1}{2} \begin{bmatrix} y_1 - \nu_1 \\ \vdots \\ y_n - \nu_n \end{bmatrix}^T D^{-1} \begin{bmatrix} y_1 - \nu_1 \\ \vdots \\ y_n - \nu_n \end{bmatrix}, \quad \text{where} \quad D = \begin{bmatrix} \Sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Sigma_n \end{bmatrix}$$

we may write formula (292) as:

$$D = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\nu})^T D^{-1} (\boldsymbol{y}-\boldsymbol{\nu})}}{\sqrt{(2\pi)^n \det D}} (\boldsymbol{y} - \boldsymbol{\nu})(\boldsymbol{y} - \boldsymbol{\nu})^T d\boldsymbol{y} \tag{295}$$

where $\boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $d\boldsymbol{y} = dy_1...dy_n$. Indeed, this is because the off-diagonal terms of the covariance matrix are 0 due to independence, while the diagonal terms are simply the variances of $Y_1, ..., Y_n$.

Now let's consider a general vector $\boldsymbol{X}$ of normal distributions $X_1, ..., X_n$. The setup here is that we consider an arbitrary $\mu \in \mathbb{R}^n$ and an arbitrary positive definite matrix S, and let:

$$p(x_1, ..., x_n) = \frac{e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T S^{-1} (\boldsymbol{x}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det S}} \tag{296}$$

Remark. *The reason why we require S to be positive definite is that the quantity:*

$$E = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T S(\boldsymbol{x} - \boldsymbol{\mu})$$

can be interpreted as the kinetic energy of some physical system of particles with velocities $x_1, ..., x_n$ (the fact that we subtract the average $\boldsymbol{\mu}$ means that we choose our coordinates so that the center of mass of the system stands still). The expression e^{-E} is called the Boltzmann factor in physics. Its presence means that velocity vectors $(x_1, ..., x_n)$ with lower energy arise with higher probability and those with higher energy arise with lower probability.

In the situation at hand, the covariance matrix (292) can be written as:

$$K = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T S^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det S}} (\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})^T d\boldsymbol{x} \tag{297}$$

To summarize: the entries of the random vector $\boldsymbol{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ whose covariance matrix is K are no longer independent. However, we can find appropriate linear combinations which are independent, by diagonalizing the matrices S and K. This is PCA for continuous probabilities, and it goes something like this. First diagonalize the positive definite matrix S that appears in (296):

$$S = QDQ^T$$

where Q is orthogonal and D is diagonal (with positive diagonal entries). Then if we perform the substitution $\boldsymbol{Y} = Q^T \boldsymbol{X} \Leftrightarrow \boldsymbol{X} = Q\boldsymbol{Y}$ (note that the mean of $\boldsymbol{Y}$ is $\boldsymbol{\nu} = Q^T \boldsymbol{\mu}$), then (297) becomes:

$$K = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\nu})^T D^{-1}(\boldsymbol{y}-\boldsymbol{\nu})}}{\sqrt{(2\pi)^n \det S}} Q(\boldsymbol{y} - \boldsymbol{\nu})(\boldsymbol{y} - \boldsymbol{\nu})^T Q^T d\boldsymbol{y} =$$

$$= Q \left[\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\nu})^T D^{-1}(\boldsymbol{y}-\boldsymbol{\nu})}}{\sqrt{(2\pi)^n \det D}} (\boldsymbol{y} - \boldsymbol{\nu})(\boldsymbol{y} - \boldsymbol{\nu})^T d\boldsymbol{y} \right] Q^T \overset{(295)}{=} QDQ^T = S$$

(we are using $\det S = \det D$ and the fact that the determinant of the orthogonal matrix Q is ± 1). Since the covariance matrix of the random vector $\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$ is diagonal, the random variables $Y_1, ..., Y_n$ are independent normal distributions. Hence under orthogonal linear transformations Q, random vectors transform as $\boldsymbol{X} = Q\boldsymbol{Y}$, and their covariance matrix transforms as $K = QDQ^T$.

The principles of least squares also find their place in probability. Suppose A is a matrix, that $b_1, ..., b_m$ are random variables with variance $\Sigma_1, ..., \Sigma_m$, and you are trying to minimize the error in the least squares approximation to the linear system:

$$A\boldsymbol{v} \sim \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \tag{298}$$

The appropriate notion of error in the experiment is:

$$\varepsilon = \sum_{i=1}^{n} \frac{(\boldsymbol{b} - A\boldsymbol{v})_i^2}{\Sigma_i} \tag{299}$$

Indeed, if all the variances $\Sigma_1, ..., \Sigma_n$ were equal, then the error would be a multiple of the square of the length of the vector $\boldsymbol{b} - A\boldsymbol{v}$, so we would be solving the ordinary least squares problem. But if the Σ_i's are different, this means that we weigh the error by the inverse of the variance: those measurements b_i with smaller (respectively larger) variance are more (respectively less) reliable, hence they will weight more (respectively less) in the error. Still, the problem of minimizing (299) is actually equivalent with the ordinary least squares problem for the related system:

$$A'\boldsymbol{v} \sim \boldsymbol{b}', \qquad \text{where} \quad A' = D^{-\frac{1}{2}}A, \quad \boldsymbol{b}' = D^{-\frac{1}{2}}\boldsymbol{b} \quad \text{and} \quad D = \begin{bmatrix} \Sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Sigma_n \end{bmatrix}$$

Therefore, the least squares solution reads:

$$\boldsymbol{v} = (A'^T A')^{-1} A'^T \boldsymbol{b}' = (A^T D^{-1} A)^{-1} A^T D^{-1} \boldsymbol{b} \tag{300}$$

In terms of the system (298), this is known as **weighted least squares**. The most general scenario is when the random variables $b_1, ..., b_n$ are not independent, and they are related by the a covariance matrix K. In this case, the appropriate analogue of the solution (300) is:

$$\boldsymbol{v} = (A^T K^{-1} A)^{-1} A^T K^{-1} \boldsymbol{b} \tag{301}$$

and it is known as **generalized least squares**. It is precisely the choice of $\boldsymbol{v}$ which minimizes the error, whose appropriate definition is:

$$\varepsilon = (\boldsymbol{b} - A\boldsymbol{v})^T K^{-1} (\boldsymbol{b} - A\boldsymbol{v})$$

Another question you might ask is what is the covariance matrix of the solution (301) in terms of the covariance matrix $K = E[\boldsymbol{b}\boldsymbol{b}^T]$. Assume that the random vector $\boldsymbol{b}$ has zero mean, to keep notations simple (it is reasonable to expect that measuring errors will have zero mean). Then:

$$E[\boldsymbol{v}\boldsymbol{v}^T] = (A^T K^{-1} A)^{-1} A^T K^{-1} \underbrace{E[\boldsymbol{b}\boldsymbol{b}^T]}_{K} K^{-1,T} A (A^T K^{-1,T} A)^{-1} =$$

$$= (A^T K^{-1} A)^{-1} A^T K^{-1,T} A (A^T K^{-1,T} A)^{-1} = (A^T K^{-1} A)^{-1}$$

I hope this has been a convincing introduction (although we have just scratched the surface) of how linear algebra is useful in probability.

Lecture 32 (*November 27*)

Statistics is the sibling of probability, but they differ slightly in their goals. While probability anticipates the likelihood of future events, statistics interprets data from past events. Let's say we are performing an experiment that measures a certain quantity. We run the experiment n times and we get the following values, called **samples**:

$$x_1, ..., x_n \tag{302}$$

The collection of samples is often called the **data set**.

Definition 29. *The **mean** of the data set* (302) *is:*

$$\mu = \frac{1}{n}(x_1 + ... + x_n) \tag{303}$$

*The **variance** of the data set* (302) *is:*

$$\Sigma = \frac{1}{n-1}\Big[(x_1 - \mu)^2 + ... + (x_n - \mu)^2\Big] \tag{304}$$

*The **standard deviation** is the square root of the variance: $\sigma = \sqrt{\Sigma}$.*

As $n \to \infty$, the quantities (303) and (304) tend to the mean and variance of the underlying probability distribution of whatever we're measuring. For example, if you're measuring a quantity which naturally obeys a normal distribution (e.g. the numeric grades in a class), when you plot sufficiently many of them (i.e. graph the function $f(x) =$ number of students who got grade x), then your plot will look like a bell curve.

Remark. *You might wonder why we're using the denominator $n-1$ in* (304) *instead of n, which might seem much more appropriate from a probability point of view. The discrepancy, namely $\frac{n}{n-1}$ is called **Bessel's correction**, and we will explain it from the point of view of linear algebra later on. It is not so significant when n is large, but it makes a difference when n is small.*

Explicitly, the mean is just measuring the average of the samples in question, while the variance is measuring how far the samples are from their mean. These notions can be presented in the language of linear algebra, by introducing the vector:

$$\boldsymbol{o} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then the mean is equal to:

$$\mu = \frac{\boldsymbol{o}^T \boldsymbol{x}}{\boldsymbol{o}\boldsymbol{o}} \tag{305}$$

and thus $\boldsymbol{o}\mu$ can be interpreted as the projection of the vector of samples $\boldsymbol{x}$ onto the vector $\boldsymbol{o}$. Meanwhile, the variance is:

$$\Sigma = \frac{||\boldsymbol{x} - \boldsymbol{o}\mu||^2}{n-1} = \frac{||P\boldsymbol{x}||^2}{n-1} \tag{306}$$

where $P = I - \frac{\boldsymbol{o}\boldsymbol{o}^T}{\boldsymbol{o}^T\boldsymbol{o}}$ is the projection matrix onto the orthogonal complement to $\boldsymbol{o}$. This orthogonal complement is an $n-1$ dimensional vector space, and indeed we have:

$$P = \begin{bmatrix} \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \ddots & \vdots \\ -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{bmatrix} \tag{307}$$

It is elementary to show that the diagonalization of P is:

$$P = QDQ^T, \quad \text{where } Q \text{ is orthogonal and } D = \begin{bmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \tag{308}$$

Therefore, if the n entries of the sample vector $\boldsymbol{x}$ are drawn from a normal probability distribution with mean 0 and variance 1, so will the entries of the vector $\boldsymbol{y} = Q^T\boldsymbol{x}$ (this is because multiplying by orthogonal matrices does not change length). But we have:

$$||P\boldsymbol{x}||^2 = ||QDQ^T\boldsymbol{x}||^2 = ||D\boldsymbol{y}||^2 = y_1^2 + ... + y_{n-1}^2 \tag{309}$$

and so the variance of (309) is $n-1$. We have just shown that the variance of the quantity $||P\boldsymbol{x}||^2$ is $\frac{n-1}{n}$ times that of the quantity $||\boldsymbol{x}||^2$, and Bessel's correction is simply a way to correct for this.

Now assume we have two sets of data: $x_1, ..., x_n$ and $y_1, ..., y_n$ (say age and height) and we want to see how correlated they are. As in the case of probability, the appropriate notion is the **covariance**:

$$\Sigma_{xy} = \frac{1}{n-1}\Big[(x_1-\mu)(y_1-\nu) + ... + (x_n-\mu)(y_n-\nu)\Big] \tag{310}$$

where μ and ν are the means of the samples $x_1, ..., x_n$ and $y_1, ..., y_n$, respectively. Note that Σ_{xx} is just the variance of the sample set $x_1, ..., x_n$, so the reason why we use the denominator $n-1$ instead of n in (310) is still Bessel's correction, as explained above. In terms of vectors:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

the covariance takes the form:

$$\boxed{\Sigma_{xy} = \frac{\boldsymbol{x}^T P\boldsymbol{y}}{n-1}} \tag{311}$$

since $P^2 = P$, $P^T = P$ (being a projection matrix) and $P\boldsymbol{x} = \begin{bmatrix} x_1-\mu \\ \vdots \\ x_n-\mu \end{bmatrix}$, $P\boldsymbol{y} = \begin{bmatrix} y_1-\nu \\ \vdots \\ y_n-\nu \end{bmatrix}$. Also:

$$|\Sigma_{xy}| \leq \sqrt{\Sigma_{xx}\Sigma_{yy}} \tag{312}$$

just like in the case of probability. The covariance is greatest (or lowest) when x and y are the most correlated (anti-correlated). If the two data sets are independent, then you would expect the covariance to be very close to 0, and actually tend to 0 as you take more and more samples $n \to \infty$.

Finally, assume we have m data sets, so let's put them in an $n \times m$ matrix:

$$\boldsymbol{A} = \begin{bmatrix} x_1 & y_1 & z_1 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ x_n & y_n & z_n & \dots \end{bmatrix}$$

If the x's, y's, z's ... have means μ, ν, ξ ..., then:

$$P\boldsymbol{A} = \begin{bmatrix} x_1-\mu & y_1-\nu & z_1-\xi & \dots \\ \vdots & \vdots & \vdots & \ddots \\ x_n-\mu & y_n-\nu & z_n-\xi & \dots \end{bmatrix} \tag{313}$$

The **covariance matrix** of the data sets is the $m \times m$ matrix defined by:

$$K = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} & \dots \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{yz} & \dots \\ \Sigma_{zz} & \Sigma_{zy} & \Sigma_{zz} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{314}$$

and is given by the formula:

$$\boxed{K = \frac{\boldsymbol{A}^T P \boldsymbol{A}}{n-1}} \tag{315}$$

To see this, just note for instance that the $(1,2)$ entry of the matrix identity (315) is precisely (311).

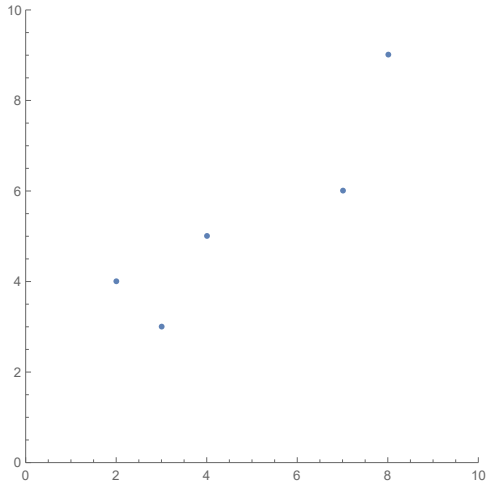
Principal component analysis (PCA) involves diagonalizing the covariance matrix. If:

$$K = QDQ^T$$

with Q orthogonal and D diagonal (with positive diagonal entries) then (315) reads:

$$QDQ^T = \frac{\boldsymbol{A}^T P \boldsymbol{A}}{n-1} \quad \Rightarrow \quad D = \frac{(\boldsymbol{A}Q)^T P (\boldsymbol{A}Q)}{n-1}$$

This means that the data set $\boldsymbol{B} = \boldsymbol{A}Q$ (whose columns are linear combinations of the constituent data sets of $\boldsymbol{A}$) has diagonal covariance. Therefore, the constituent data sets of $\boldsymbol{B}$ are mutually independent. Moreover, if the i–th diagonal entry of D is the largest (respectively smallest) then the i–th constituent data set of $\boldsymbol{B}$ has the largest (respectively smallest) variance. Therefore, this approach allows us to isolate those linear combinations in a collection of data sets which produce the most variance from those which produce the least variance. For example, consider the plot:



of two data sets: the entries of $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}$ are on the horizontal axis and the entries of $\boldsymbol{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}$

are on the vertical axis. Explicitly, they are given by

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{y} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 6 \\ 9 \end{bmatrix}$$

and suppose we wish to compute their covariance matrix, and to apply PCA to them (i.e. to find linear combinations of the data sets which are independent, and to compute the variance of those). Start by constructing the matrix:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 3 \\ 4 & 5 \\ 7 & 6 \\ 8 & 9 \end{bmatrix}$$

Then compute:

$$P\boldsymbol{A} = \frac{1}{5} \begin{bmatrix} -14 & -7 \\ -9 & -12 \\ -4 & -2 \\ 11 & 3 \\ 16 & 18 \end{bmatrix} \tag{316}$$

Then the covariance matrix is given by formula (315):

$$K = \frac{1}{20} \begin{bmatrix} 134 & 107 \\ 107 & 106 \end{bmatrix}$$

You could perform PCA by diagonalizing the matrix K, or you could instead compute the SVD of the matrix $P\boldsymbol{A}$. Let's opt for the latter:

$$P\boldsymbol{A} = U\Sigma V^T \qquad \text{where} \qquad \Sigma = \begin{bmatrix} \sqrt{24 + \frac{1}{5}\sqrt{11645}} & 0 \\ 0 & \sqrt{24 - \frac{1}{5}\sqrt{11645}} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and U and V are orthogonal matrices (but their entries are really big expressions involving square roots; you can use any programming language with a "Singular Value Decomposition" function if you really want to compute them). Then the covariance matrix is given by:

$$K = \frac{V\Sigma^T U^T U \Sigma V^T}{5-1} = V \begin{bmatrix} 6 + \frac{1}{20}\sqrt{11645} & 0 \\ 0 & 6 - \frac{1}{20}\sqrt{11645} \end{bmatrix} V^T \tag{317}$$

The meaning of this is the following: the entries of the 2×2 orthogonal matrix V teach you how to construct mutually independent combinations of the data sets $\boldsymbol{x}$ and $\boldsymbol{y}$. Explicitly, if:

$$V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

then the data sets:

$$\begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} V \qquad \Leftrightarrow \qquad \begin{cases} \boldsymbol{x}' = v_{11}\boldsymbol{x} + v_{21}\boldsymbol{y} \\ \boldsymbol{y}' = v_{12}\boldsymbol{x} + v_{22}\boldsymbol{y} \end{cases}$$

are mutually independent. Moreover, the variances of $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are the two numbers on the diagonal of (317), and numerically they are roughly equal to 11.39 and 0.6. This implies that the linear combination $\boldsymbol{y}'$ of our two data sets has very little variance, and so it is roughly constant. Meanwhile, the linear combination $\boldsymbol{x}'$ is very "noisy", since it has high variance.

Lecture 33 (*December 2*) *Reading: section 10.3*

A particular kind of matrix that arises in the study of probability is the following.

Definition 30. *A **Markov matrix** (or "stochastic matrix") is a square matrix with all entries positive, such that the entries on every column add up to 1.*

The notion above is sometimes called a "left Markov matrix", to contrast it with the transposed notion of "right Markov matrix", wherein the entries on every row add up to 1. One of the main application of Markov matrices is the following setup, called a **Markov process**. Suppose we have a system which can be described as initially occupying a finite number of states $s_1, ..., s_n$ with probabilities $v_1, ..., v_n$. Suppose that our system changes once every second [4] and that the probability of moving from state j to state i from one second to the next is a number $p_{ij} > 0$. Let's put these numbers in a matrix:

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \tag{318}$$

and note that P is indeed a Markov matrix: the sum of the entries on its j–th column is the probability of moving from state j to some other state from one second to the next, which has to be 1. So what is the probability that we find ourselves in state i after one second? Well, since initially we were in state 1 with probability v_1, ..., in state n with probability v_n, then:

$$\Big(\text{probability of state } i \text{ after one second}\Big) = p_{i1}v_1 + ... + p_{in}v_n$$

This just says that if $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ records the probabilities of the n states at initial time, then:

$$P\boldsymbol{v} = \begin{bmatrix} p_{11}v_1 + ... + p_{1n}v_n \\ \vdots \\ p_{n1}v_1 + ... + p_{nn}v_n \end{bmatrix}$$

records the probabilities of the n states after one second. Going further, this implies that:

$$P^2\boldsymbol{v}, P^3\boldsymbol{v}, ..., P^k\boldsymbol{v}, ...$$

[4] Or minute, or hour, or day etc, but let's work with "second" just to have a consistent convention

record the probabilities of the n states after 2,3,..., k,... seconds. So repeated applications of the Markov matrix P to the initial probabilities of the states reveals the probabilities of the states after any amount of time. Then a very interesting question is what happens to the system as time goes to ∞? Fortunately, we know that a good way to compute powers of a matrix is to diagonalize it:

$$P = V \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} V^{-1}$$

If $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are the eigenvectors of P (i.e. the columns of V) then we have the identity:

$$\text{if } \boldsymbol{v} = c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n \qquad \Rightarrow \qquad P^k\boldsymbol{v} = c_1\lambda_1^k\boldsymbol{v}_1 + ... + c_n\lambda_n^k\boldsymbol{v}_n \tag{319}$$

So to compute the long-term behavior of the system, write the initial vector of probabilities $\boldsymbol{v}$ as a linear combination of the eigenvectors of P, and then just apply (319) for any k. But now let's do some qualitative analysis. As a consequence of (319), the dominant rate of growth of $P^k\boldsymbol{v}$ is comparable to the k–th power of its largest eigenvalue.

- if P has an eigenvalue > 1 (in absolute value) then some entries of $P^k\boldsymbol{v}$ would tend to ∞
- if all the eigenvalues of P are < 1 (in absolute value), then $P^k\boldsymbol{v}$ would tend to 0

None of the two options above make any sense, because the vector $P^k\boldsymbol{v}$ indicates the probabilities of the states at time $t = k$ seconds; as such, its entries are positive numbers which add up to 1. So it seems that the only thing which makes qualitative sense is if P had some eigenvalues of size 1, and all the other eigenvalues were smaller than 1. Indeed, this is what happens due to the following fact, known as the Perron-Frobenius Theorem:

Fact 22. *If a square matrix A has positive entries, then it has a unique largest eigenvalue $\lambda_{\max} > 0$. This eigenvalue is real, and all other eigenvalues are smaller than $\lambda_{\max}$ in absolute value.*

The eigenvalue $\lambda_{\max}$ has algebraic and geometric multiplicity one; the unique (up to scalar) eigenvector can be chosen to have all positive entries.

Moreover, in the Markov matrix case (all column sums are 1) the largest eigenvalue is $\lambda_{\max} = 1$. If we let $\boldsymbol{v}_{\max}$ be the corresponding eigenvector, then we have:

$$P\boldsymbol{v}_{\max} = \boldsymbol{v}_{\max}$$

hence $\boldsymbol{v}_{\max}$ is a **stationary** vector of probabilities: it is unchanged by the evolution of our system. Moreover, formula (319) and Fact 22 show that any initial vector of probabilities $\boldsymbol{v}$ approaches the unique stationary vector $\boldsymbol{v}_{\max}$ as time $k \to \infty$.

Let's give an example of a Markov matrix: there are 5 boxes in a row, and initially there's a cat in box 1 and a mouse in box 5. Every second, each animal moves to an adjacent box with equal

probability (if the animal is in one of the outermost boxes, they must jump to the next one inward). The process ends when the cat and mouse find themselves in the same box, at which point the cat eats the mouse. To describe this game mathematically, we consider the various states of the system to be the combinations (cat's box, mouse's box):

- state 1: the cat is in box 1 and the mouse is in box 3
- state 2: the cat is in box 1 and the mouse is in box 5
- state 3: the cat is in box 2 and the mouse is in box 4
- state 4: the cat is in box 3 and the mouse is in box 5
- state 5: the cat and mouse are in the same box

You may ask where are all the other combinations of boxes. First of all, any jump preserves the sum of parities of the animals' boxes. Since the initial state is $(1, 5)$, the animals' boxes must always be (odd,odd) or (even,even): they must always be an even number of boxes away from each other. Secondly, there's no reason to consider the cases when the cat's box is to the right of the mouse's box, because they can't get to such a situation without the cat eating the mouse first. In terms of the 5 states above, the Markov matrix that records the probabilities of going from one state to another is:

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

The initial condition is $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, but the long-term behavior of the system approaches $\boldsymbol{v}_{\max} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, namely the eigenvector of P corresponding to the largest eigenvalue $\lambda_{\max} = 1$. This means that, as time goes to ∞, the cat is almost certain in probability to catch the mouse (in fact, a finer analysis shows that the mouse's expected time of survival is just 4.5 seconds).

A related subject is the study of population growth. This is a situation where you have a number of species sharing an environment (say rabbits and wolves). You can put their values in a vector:

$$\begin{bmatrix} r_0 \\ w_0 \end{bmatrix}$$

where r_n is the number of rabbits and w_n is the number of wolves at the end of year n. You are interested in the evolution of these populations from one year to the next. We assume the model

of population growth obeys a linear evolution formula, i.e.:

$$\begin{bmatrix} r_k \\ w_k \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r_{k-1} \\ w_{k-1} \end{bmatrix}$$

(this is not very realistic, but in practice this can serve as an approximation for more complicated models). The goal is to say something about the long term evolution of the populations. It's pretty straightforward to see that:

$$\begin{bmatrix} r_k \\ w_k \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}^k \begin{bmatrix} r_0 \\ w_0 \end{bmatrix}$$

To raise the matrix above to the k–th power, we must diagonalize it, which is pretty straightforward:

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = VDV^{-1}$$

where:

$$V = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Then the long-term growth of the populations is given by:

$$\begin{bmatrix} r_k \\ w_k \end{bmatrix} = V \begin{bmatrix} 3^k & 0 \\ 0 & 2^k \end{bmatrix} V^{-1} \begin{bmatrix} r_0 \\ w_0 \end{bmatrix}$$

The conclusion is that, up to linear combinations, the number of rabbits and wolves grows exponentially (and this rate of growth does not care about the initial condition).

Lecture 34 (*December 4*) *Reading: section 10.1*

One of the basic notions in combinatorics and computer science is graphs.

Definition 31. *A **graph** is a collection of **vertices** (or nodes) connected by **edges**.*

It is important to stress that the geometric location of the vertices and the shape of the edges is not important when dealing with graphs. All that matters is the abstract notion that certain pairs of vertices are connected. Therefore, a graph is encoded by the following data:

- The set $\{1, .., n\}$ (which are simply labels for the vertices)
- A certain set of pairs (i, j) for various $i \neq j$ among $\{1, ..., n\}$ (the edges)

There are generalizations of graphs, which allow edges (i, i) (called "loops") and which also allow multiple edges between any pair (i, j) of vertices, but we will not be studying these in this class. However, we will assume that all of our edges are oriented, i.e. they go "from" vertex i "to" vertex j.

Examples include cities (vertices) connected by roads (edges), communication nodes (vertices) connected by wires (edges) and so on. To any graph, we may associate the following matrices:

$$\text{the } n \times n \textbf{ adjacency} \text{ matrix } A$$

$$\text{the } n \times m \textbf{ incidence} \text{ matrix } B$$

where n is the number of vertices and m is the number of edges. By definition:

$$A_{ij} = \begin{cases} 1 & \text{if vertices } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases} \tag{320}$$

$$B_{ik} = \begin{cases} -1 & \text{if edge } k \text{ comes out of vertex } i \\ 1 & \text{if edge } k \text{ goes into vertex } i \\ 0 & \text{otherwise} \end{cases} \tag{321}$$

The latter notation presupposes a labeling of the edges with the numbers $1, ..., m$. Therefore, the matrix B has a single 1, a single -1 and the rest 0's on every column. We may also define the $n \times n$ **Laplacian** matrix by

$$L_{ij} = \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if vertices } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

where d_i is the **degree** of the vertex i, namely the number of edges at that vertex. We have:

$$L = BB^T \qquad \text{and} \qquad L = \text{diagonal matrix} - A \tag{322}$$

which shows the fact that both L and A are symmetric.

Example 6. *Consider the* $(2, 3)$ *complete bipartite graph: it has* $2 + 3 = 5$ *vertices, and* $2 \cdot 3 = 6$ *edges: there is an edge from any one of the vertices* $\{1, 2\}$ *into any one of the vertices* $3, 4, 5$*. Its adjacency matrix is:*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

its Laplacian matrix is:

$$L = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

and its incidence matrix is:

$$B = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

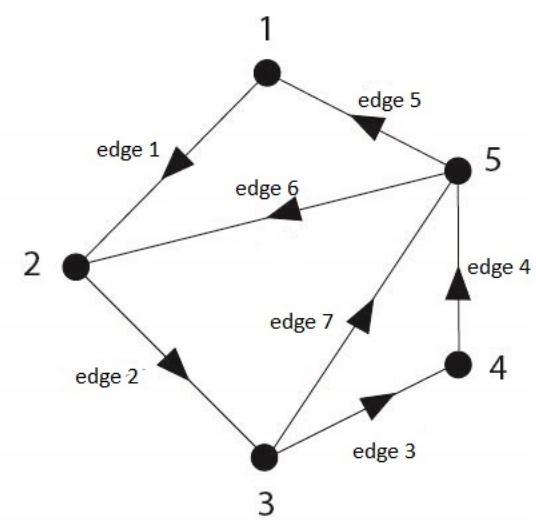
The following facts show how the matrices A, B, L encode various things about the graph.

Fact 23. *The (i, j) entry of A^k is the number of length k paths starting at i and ending at j.* [5].

Fact 24. *The rank of B is n minus the number of connected components of the graph.* [6]

Fact 25. *All cofactors of the matrix L are equal to the number of spanning trees in the graph.* [7]

We have already encountered graphs in the setting of electric circuits. Take for example:



where the edges are conducting wires connecting certain pairs of nodes (the vertices), and the orientation of an edge dictates which direction the current is flowing. The incidence matrix of this graph is:

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

We are interested in the vector of currents passing through each of the wires:

$$\boldsymbol{w} = \begin{bmatrix} I_1 \\ \vdots \\ I_7 \end{bmatrix}$$

[5] A path of length k is a sequence of k contiguous edges which start at vertex i and end at vertex j. If we wanted to only count oriented paths, i.e. sequences of edges where we always move in the direction of the arrows, then we would need to replace the adjacency matrix A by a related matrix. Try and guess what that matrix would have to be

[6] A connected component is a subset $S \subset \{1, ..., n\}$ of vertices such that any $i, j \in S$ are connected by a path, and no $i \in S, j \notin S$ are connected by an edge

[7] A spanning three is a collection of $n - 1$ edges which connect all the n vertices of our graph

(we write 7 for the number of wires to keep in line with the example pictured on the previous page). Then Kirchhoff's law for currents states that, at each node in the circuit, the total current in is equal to the total current out. In terms of matrices, this reads:

$$\textbf{Kirchhoff's law:} \qquad B\boldsymbol{w} = 0 \tag{323}$$

Therefore, the vector of currents $\boldsymbol{w}$ is constrained to lie in the nullspace of B. (more generally, you may put a non-zero vector in the right-hand side of (323), signifying current sources/sinks being placed at the nodes). Another important vector of our circuit consists of voltages at the nodes:

$$\boldsymbol{v} = \begin{bmatrix} V_1 \\ \vdots \\ V_4 \end{bmatrix}$$

The entries of the vector $B^T\boldsymbol{v}$ measure the voltage drops across the 7 wires in the circuit. Assuming all the wires have the same resistance R (e.g. they are made from the same material), then Ohm's law states that the voltage drop across every wire is equal to R times the current across that wire. As a vector equation, this reads:

$$\textbf{Ohm's law:} \qquad B^T\boldsymbol{v} = R\boldsymbol{w} \tag{324}$$

If you combine (323) and (324), you arrive at:

$$BB^T\boldsymbol{v} = 0 \tag{325}$$

But now recall (322): $L = BB^T$ is the Laplacian matrix of the grap, so $\boldsymbol{v} \in N(L)$.

Remark. *In real-life circuits, the equation* (325) *is replaced by:*

$$BCB^T\boldsymbol{v} = \boldsymbol{a}$$

where C is a diagonal matrix which keeps track of the conductance (inverse resistance) of each wire in our circuit, and $\boldsymbol{a}$ is the vector keeping track of the voltages of outside power sources at each node.

Lecture 35 (*December 9*) *Reading: section 10.5*

We will now study Fourier series, which is a very useful application of linear algebra in analysis, and from there in applied mathematics and engineering. But the first difference is that one does not study finite-dimensional vector spaces, but infinite-dimensional ones.

Definition 32. *Let V be the vector space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with period 2π, i.e.:*

$$f(x + 2\pi) = f(x) \qquad \forall x \in \mathbb{R} \tag{326}$$

Such functions are called ***periodic****.*

The fact that we only look at periodic functions might seem like a restriction, but it's suitable for many applications (think signals, waves). Also, the fact that the period is 2π rather than any other number is not a meaningful restriction, but more of a notational convention. The reason for this is that if $f(x)$ is a function with period L, then $f(\frac{2\pi x}{L})$ is a function with period 2π.

The fact that the set of f's as in (326) is a vector space is easy to see, basically because any linear combination of differentiable periodic functions is differentiable and periodic (this entails the fact that all of these functions have the same period, but we have already assumed that this is the case). To convince you that it is not a finite-dimensional vector space, consider the functions:

$$1, \cos x, \cos(2x), \cos(3x), ..., \sin x, \sin(2x), \sin(3x), ... \tag{327}$$

We claim that all of the infinitely many functions in (327) are linearly independent: there is no linear combination of these functions which is equal to the zero function [8]. Therefore, the space V is an entirely new beast from the vector spaces $\mathbb{R}^n$ that we have studied in most of our course. However, it does share a lot of features with finite-dimensional vector spaces, such as the existence of a dot product like operation, the **inner product** of functions:

$$(f, g) = \int_{-\pi}^{\pi} f(x)g(x)dx \tag{328}$$

[9] In other words, the inner product takes two functions and produces a number, just like the dot product took two vectors and produced a number. And just like the length of a vector is its dot product with itself, we have the **norm** of a function:

$$||f||^2 = (f, f) = \int_{-\pi}^{\pi} f(x)^2 dx \tag{329}$$

which is a positive real number (unless $f = 0$, in which case the norm is 0). Here's one important fact: the functions (327) are all mutually orthogonal with respect to the inner product (328):

$$(\sin kx, \sin lx) = 0, \quad \text{unless } k = l, \text{ in which case } (\sin kx, \sin kx) = \pi \tag{330}$$

$$(\cos kx, \cos lx) = 0, \quad \text{unless } k = l, \text{ in which case } (\cos kx, \cos kx) = \pi \tag{331}$$

$$(\sin kx, \cos lx) = 0, \quad \text{for all } k, l \tag{332}$$

The formulas above also hold if k or l are 0 (except that $\sin 0 = 0$ and $\cos 0 = 1$ imply that $(0, 0) = 0$ and $(1, 1) = 2\pi$). So (327) is a collection of linearly independent, mutually orthogonal elements of the vector space V. If this were a finite-dimensional vector space, we could hope that the collection (327) is also a basis. Well, something pretty close to this actually happens:

Fact 26. *Any function $f \in V$ can be written as a linear combination of the functions* (327)*, but the linear combination has to be infinite. In other words, we can write any function $f \in V$ as:*

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + ... + b_1 \sin x + b_2 \sin 2x + ... \tag{333}$$

*The numbers $a_0, a_1, a_2, ..., b_1, b_2, ...$ are called the **Fourier coefficients** of $f(x)$, and the sum* (333) *is called the **Fourier series** of $f(x)$. It is a theorem that the Fourier series converges.*

[8] Note the emphasis on the word "linear", since there are non-linear combinations of the above functions which are equal to the zero function, for example $(\sin x)^2 + (\cos x)^2 - 1 = 0$. However, when studying vector spaces, all we are concerned with are linear combinations

[9] The fact that the integral goes from $-\pi$ to π is not important. You could have taken it over any interval of length 2π and the value of the integral would have been unchanged, due to the periodicity of the functions f and g

In signal processing (and I guess, in music) the functions $\cos kx, \sin kx$ are called **harmonics**, so the Fourier series is basically just a way to decompose an arbitrary periodic signal $f(x)$ into harmonics. As you would expect, in real-life examples the coefficients a_n, b_n tend to 0 as $n \to \infty$, so only the first few harmonics of a signal dominate it.

Linear algebra also gives us a very practical way to compute the Fourier coefficients of any function. Because the functions (327) are orthogonal, you can compute these coefficients by computing the inner product of f with any one of these functions:

$$(f, 1) = a_0(1, 1) \qquad \Rightarrow \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \tag{334}$$

$$(f, \cos kx) = a_k(\cos kx, \cos kx) \quad \Rightarrow \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)(\cos kx)dx \tag{335}$$

$$(f, \sin kx) = a_k(\sin kx, \sin kx) \quad \Rightarrow \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)(\sin kx)dx \tag{336}$$

Formulas (334), (335), (336) show that if you have a machine that computes various integrals of f (and there are many ways for getting approximate answers) this immediately gives a recipe for computing the Fourier series of f. Here's an example of this setup. Consider the function:

$$f(x) = x \quad \text{for all } x \in [-\pi, \pi] \tag{337}$$

then extended to all real numbers x so that it has period 2π (note that this function has a discontinuity at $x = \pi$; all the facts in this class still hold in this slightly more general setup). To compute its Fourier series, we need to compute the following integrals:

$$(f, \cos kx) = \int_{-\pi}^{\pi} x(\cos kx)dx = 0$$
$$(f, \sin kx) = \int_{-\pi}^{\pi} x(\sin kx)dx = (-1)^{k-1}\frac{2\pi}{k}$$

which can be done e.g. by integration by parts. Therefore (334)–(336) imply the Fourier series:

$$f(x) = \frac{2\sin x}{1} - \frac{2\sin 2x}{2} + \frac{2\sin 3x}{3} - \frac{2\sin 4x}{4} + ... \tag{338}$$

The fact that the Fourier series only contains sines in this example should not be surprising; after all, the particular function $f(x)$ in (337) is odd. But then let us compute the norm squared of the function (337). On one hand, we have:

$$(f, f) = \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3}$$

On the other hand, the norm squared of the right-hand side of (338) is easy to compute, because the functions $\sin kx$ have pairwise zero inner product:

$$(f, f) = \frac{4\pi}{1^2} + \frac{4\pi}{2^2} + \frac{4\pi}{3^2} + \frac{4\pi}{4^2} + ...$$

By equating the two expressions above, we arrive at the interesting identity:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$$

This (and others like it) is a neat identity involving π, which gives an easy way to approximate the powers of π. Based on the formula above, you could even approximate π using pen and paper (assuming you remember the algorithm for taking square roots by hand).

Let's now discuss a variant of Fourier series, wherein we replace the vector space V from Definition 32 by the vector space of differentiable functions with complex (instead of real) values:

$$f : \mathbb{R} \to \mathbb{C}$$

with period 2π. Because complex numbers are of the form $a + bi$ where a, b are real numbers, such complex-valued functions are of the form $a(x) + b(x)i$ where $a(x), b(x)$ are real-valued functions. Much of the theory concerning Fourier series carries through in the present setup, but in a sense it's simpler and more elegant. This is because instead of working with the particular real-valued functions (327), you can work with the complex-valued functions:

$$e^{ikx} \quad \text{for all integers } k \tag{339}$$

In fact, the functions in (327) are merely the real/imaginary parts of the functions in (339). In the complex-valued setting, the definition of inner product and norm from (328) should be changed to:

$$(f, g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx \tag{340}$$

where $\overline{z}$ denotes the complex conjugate of the complex number z. The reason for this is that you want the norm of any function to be a non-negative real number:

$$||f||^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx \geq 0 \tag{341}$$

and you would not have been able to achieve this without the conjugate in (340) (what I'm saying is that if z is an arbitrary complex number, z^2 is also a complex number, but $z\overline{z} = |z|^2$ is a non-negative real number). It is then easy to compute the inner products of the functions (339):

$$(e^{ikx}, e^{ilx}) = \int_{-\pi}^{\pi} e^{ikx}\overline{e^{ilx}}dx = \int_{-\pi}^{\pi} e^{i(k-l)x}dx = \begin{cases} \frac{e^{i(k-l)x}}{i(k-l)}\Big|_{-\pi}^{\pi} = 0 & \text{if } k \neq l \\ 2\pi & \text{if } k = l \end{cases}$$

So the functions (339) are pairwise orthogonal. The complex-valued analogue of Fact 26 is that any differentiable complex-valued periodic function has a Fourier series of the form:

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx} \tag{342}$$

where the Fourier coefficients are given by:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx \tag{343}$$

The advantage of complex-valued Fourier series over real-valued ones is clear: the Fourier series (342) is a sum of only one type of function, as opposed from two types of functions in (333). Moreover, you only need to remember the single formula (343) for the Fourier coefficients in the complex-valued case, instead of formulas (334), (335), (336) in the real-valued case. Moreover, since any real-valued function can also be thought of as a complex valued function, it still has a Fourier series of the form (342) (it's just that you will have $\overline{c_k} = c_{-k}$, in order for the imaginary parts of various complex exponentials to cancel each other out).

Lecture 36 (*December 11)*

In the previous lecture, we studied the infinite-dimensional vector space of functions (and their Fourier series, which were infinite sums) because it was the first to arise historically. But there is a closely related version, which gives rise to a finite-dimensional space, which we will now study. This newer version also stems from signal processing: when receiving a signal whose shape is a periodic function $f(x)$, you often don't know the formula for the whole function. All you can measure are the values of this function at finitely many times, and it makes sense to do so at equally spaced times. So you could approximate the signal by knowing the values:

$$v_1 = f\left(\frac{2\pi}{N}\right), v_2 = f\left(\frac{4\pi}{N}\right), ..., v_N = f\left(2\pi\right)$$

for some large natural number N. Therefore, instead of the function $f(x)$, we record the vector:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \tag{344}$$

The vector above lives in the **physical space** $\mathbb{R}^N$. So by only recording the values of f at N points along the real line, we have been able to reduce from the infinite-dimensional vector space V where f lives to the finite-dimensional physical space $\mathbb{R}^N$ where $\boldsymbol{v}$ lives. But now we recall that f also has a Fourier series, and in fact we assume that it is only a sum of the first N harmonics:

$$f(x) = c_1 e^{ix} + ... + c_N e^{iNx} \tag{345}$$

[10]. Then the function $f(x)$ can also be determined by the vector of its Fourier coefficients:

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} \tag{346}$$

The vector $\boldsymbol{c}$ lives in **frequency space** $\mathbb{R}^N$. Even though it has the same dimension as the physical space where (344) lives, it is a bit more intuitive to think of physical and frequency spaces as being two completely different things. Of course, it is a very good question of how to go between these two spaces. Plugging in the values $x = \frac{2\pi k}{N}$ into (345) leads to the equality:

$$v_k = \sum_{l=1}^{N} c_l e^{\frac{2\pi i k l}{N}} \tag{347}$$

Let us consider the N–th order root of unity (see (228)):

$$\omega = e^{\frac{2\pi i}{N}} \tag{348}$$

Then (347) can be rewritten as:

$$\boxed{\boldsymbol{v} = \boldsymbol{F}\boldsymbol{c}} \tag{349}$$

[10]This is not such a far-fetched restriction, since the Fourier coefficients c_k tend to 0 as $k \to \infty$; the fact that $f(x)$ has no c_0 term just means that it has integral 0, i.e. we have subtracted its average

for the $N \times N$ matrix whose (k, l) coefficient is ω^{kl}:

$$\boldsymbol{F} = \begin{bmatrix} \omega^{1\cdot 1} & \omega^{1\cdot 2} & \dots & \omega^{1\cdot(N-1)} & \omega^{1\cdot N} \\ \omega^{2\cdot 1} & \omega^{2\cdot 2} & \dots & \omega^{2\cdot(N-1)} & \omega^{2\cdot N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega^{(N-1)\cdot 1} & \omega^{(N-1)\cdot 2} & \dots & \omega^{(N-1)\cdot(N-1)} & \omega^{(N-1)\cdot N} \\ \omega^{N\cdot 1} & \omega^{N\cdot 2} & \dots & \omega^{N\cdot(N-1)} & \omega^{N\cdot N} \end{bmatrix} \tag{350}$$

This matrix has the property that:

$$\overline{\boldsymbol{F}}\boldsymbol{F} = N \cdot I \qquad \Rightarrow \qquad \boldsymbol{F}^{-1} = \frac{\overline{\boldsymbol{F}}}{N} \tag{351}$$

hence the inverse $\boldsymbol{F}^{-1}$ is a very similar matrix to (350):

$$\boldsymbol{F}^{-1} = \frac{1}{N} \begin{bmatrix} \omega^{-1\cdot 1} & \omega^{-1\cdot 2} & \dots & \omega^{-1\cdot(N-1)} & \omega^{-1\cdot N} \\ \omega^{-2\cdot 1} & \omega^{-2\cdot 2} & \dots & \omega^{-2\cdot(N-1)} & \omega^{-2\cdot N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega^{-(N-1)\cdot 1} & \omega^{-(N-1)\cdot 2} & \dots & \omega^{-(N-1)\cdot(N-1)} & \omega^{-(N-1)\cdot N} \\ \omega^{-N\cdot 1} & \omega^{-N\cdot 2} & \dots & \omega^{-N\cdot(N-1)} & \omega^{-N\cdot N} \end{bmatrix} \tag{352}$$

(here we used the fact that $\overline{\omega} = \omega^{-1} = e^{-\frac{2\pi i}{N}}$). Therefore, property (349) is equivalent to:

$$\boxed{\boldsymbol{c} = \boldsymbol{F}^{-1}\boldsymbol{v}} \tag{353}$$

Formulas (349) and (353) take you from physical space to frequency space and back. In particular, the matrix $\boldsymbol{F}^{-1}$ is traditionally called the **Fourier transform**, but the same name could be given to the matrix $\boldsymbol{F}$. This is because formula (351) implies that taking the inverse of this matrix amounts to taking the conjugates of its entries and dividing by N, both pretty harmless operations.

Being able to represent a signal in both physical ((344)) and frequency ((346)) form is important because either of these forms is good for some things and bad for others. For example, physical space is good when multiplying signals:

$$\begin{aligned} &\text{if } f(x) \text{ corresponds to } v_k = f\left(\frac{2\pi k}{N}\right) \text{ in physical space, and} \\ &\text{if } g(x) \text{ corresponds to } w_k = g\left(\frac{2\pi k}{N}\right) \text{ in physical space, then} \\ &\text{then } f(x)g(x) \text{ corresponds to } v_k w_k = f\left(\frac{2\pi k}{N}\right) g\left(\frac{2\pi k}{N}\right) \end{aligned}$$

So the physical coordinates of a product of functions are just the products of the physical coordinates of the two functions. Meanwhile, frequency space is good for understanding **convolution** of signals. In the world of differentiable functions $f(x)$ and $g(x)$, their convolution is defined as the function:

$$f * g = h \qquad \text{where} \qquad h(x) = \int_{-\pi}^{\pi} f(y)g(x-y)dy \tag{354}$$

In our discrete world, where we approximate the functions f and g by their values at the multiples of $\frac{2\pi}{N}$ (i.e. their physical space coordinates), we have:

$$\boldsymbol{v} * \boldsymbol{v}' := \text{the vector } \boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \quad \text{such that} \quad w_k = \frac{1}{N} \sum_{l=1}^{N} v_l v'_{k-l} \tag{355}$$

(the indices of v' are taken with period N, i.e. we write $v'_l = v'_{N+l}$ if $l \leq 0$). In this language, the frequency space coordinates of the convolution are simply the products of the frequency space coordinates of $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$\boldsymbol{F}^{-1}\boldsymbol{w} = \text{coordinate-wise multiplication of } (\boldsymbol{F}^{-1}\boldsymbol{v}) \text{ and } (\boldsymbol{F}^{-1}\boldsymbol{v}') \tag{356}$$

So it is very helpful to have an efficient way to go between the physical and frequency coordinates of a signal. You might think that formulas (349) and (353) are enough: after all, you only need to multiply an $N \times N$ matrix with a vector. But the number of operation this takes is of the order of N^2, which is huge when the number N measures in the thousands. Fortunately, people have come up with a great algorithm called the **Fast Fourier Transform (FFT)** for multiplication with $\boldsymbol{F}$, which only takes approximately $N \log N$ steps. The way to do this is to note that products such as (349) would take significantly fewer than N^2 steps if the matrix $\boldsymbol{F}$ were sparse, i.e. had a lot of zeroes. This is alas not the case, but not all is lost, since $\boldsymbol{F}$ can be written as a product of sparser matrices:

$$\boldsymbol{F}_{2N} = \begin{bmatrix} I_N & D_N \\ I_N & D_N \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_N & 0 \\ 0 & \boldsymbol{F}_N \end{bmatrix} P_{2N} \tag{357}$$

where:

- $\boldsymbol{F}_N$ is the $N \times N$ matrix (350)
- $\boldsymbol{F}_{2N}$ is the analogous matrix obtained by starting with $2N$ instead of N
- D_N is the diagonal matrix with entries $\omega, \omega^2, ..., \omega^N$
- P_{2N} is the permutation matrix corresponding to the permutation $(1, 3, ..., 2N-1, 2, 4, ..., 2N)$

Since the first and third matrices in the right-hand side of (357) are very sparse, they contribute little to the time it takes to multiply the matrix $\boldsymbol{F}_{2N}$ to any vector. Meanwhile, the second matrix in the right-hand side of (357) is half-full of zeroes. Iterating the above procedure (i.e. writing the matrix $\boldsymbol{F}_{2N}$ in terms of $\boldsymbol{F}_N$ via (357), then in terms of $\boldsymbol{F}_{N/2}$ etc...) allows you to compute (349) in $N \log N$ time. By conjugation, the same is true of (353).